GammaLib Maths

Mathematical Implementation

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1 Introduction

2 Functions

2.1 King profile

The King Profile is a commonly used parametrisation of instrument PSFs in astronomy. It is radially symmetric and compared to a simple Gaussian, it allows longer tails in the distribution of events from a point source. The probability density function is defined as follows:

$$P(r|\sigma,\gamma) = \frac{1}{2\pi\sigma^2}\left(1-\frac{1}{\gamma}\right)\left(1+\frac{1}{2\gamma}\frac{r^2}{\sigma^2}\right)^{-\gamma} \tag{1}$$

To integrate the function we substitute

$$u = \frac{r^2}{2\sigma^2} \tag{2}$$

which gives

$$rdr = \sigma^2 du \tag{3}$$

and hence

$$\frac{1}{2\pi\sigma^2}\left(1-\frac{1}{\gamma}\right)\left(1+\frac{1}{2\gamma}\frac{r^2}{\sigma^2}\right)^{-\gamma} rdr = \frac{1}{2\pi\sigma^2}\left(1-\frac{1}{\gamma}\right)\left(1+\frac{u}{\gamma}\right)^{-\gamma}\sigma^2 du \tag{4}$$

The integral is thus given by

$$\begin{aligned}
\int_0^{r_{\max}}\int_0^{2\pi} P(r|\sigma,\gamma) rdrd\phi &= \left(1-\frac{1}{\gamma}\right)\int_0^{u_{\max}}\left(1+\frac{u}{\gamma}\right)^{-\gamma} du \\
&= \left(1-\frac{1}{\gamma}\right)\left(\frac{(\gamma+u)\left(\frac{\gamma+u}{\gamma}\right)^{-\gamma}}{1-\gamma}\right)\Bigg|_0^{u_{\max}} \\
&= \left(\frac{\gamma-1}{\gamma}\right)\left(\frac{(\gamma+u)\left(\frac{\gamma+u}{\gamma}\right)^{-\gamma}}{1-\gamma}\right)\Bigg|_0^{u_{\max}} \\
&= -\left(\frac{(\gamma+u)\left(\frac{\gamma+u}{\gamma}\right)^{-\gamma}}{\gamma}\right)\Bigg|_0^{u_{\max}} \\
&= -\left(\frac{\gamma+u}{\gamma}\right)^{1-\gamma}\Bigg|_0^{u_{\max}} \\
&= -\left(1+\frac{u}{\gamma}\right)^{1-\gamma}\Bigg|_0^{u_{\max}} \\
&= -\left(1+\frac{u_{\max}}{\gamma}\right)^{1-\gamma}+1 \\
&= 1-\left(1+\frac{u_{\max}}{\gamma}\right)^{1-\gamma}
\end{aligned} \tag{5}$$

where

$$u_{\max} = \frac{r_{\max}^2}{2\sigma^2} \tag{6}$$

To determine the value of $u_{\rm max}$ that corresponds to a given containment fraction F, the equation

$$1 - \left(1 + \frac{u_{\rm max}}{\gamma}\right)^{1-\gamma} = F \tag{7}$$

has to be solved for $u_{\rm max}$, resulting in

$$u_{\rm max} = \left((1-F)^{\frac{1}{1-\gamma}} - 1\right)\gamma \tag{8}$$

This can be converted into $r_{\rm max}$ using

$$r_{\rm max} = \sigma\sqrt{2u_{\rm max}} \tag{9}$$

3 Spatial models

3.1 ShellFunction

3.1.1 Small angle approximation

The shell function is a radial function $f(\theta)$, where θ is the angular separation between shell centre and the actual location. In the small angle approximation $\sin\theta \approx \theta$, the shell function is given by

$$f(\theta) = f_0 \begin{cases} \sqrt{\theta_{\rm out}^2 - \theta^2} - \sqrt{\theta_{\rm in}^2 - \theta^2} & \text{if } \theta \leq \theta_{\rm in} \\ \sqrt{\theta_{\rm out}^2 - \theta^2} & \text{if } \theta_{\rm in} < \theta \leq \theta_{\rm out} \\ 0 & \text{if } \theta > \theta_{\rm out} \end{cases} \tag{10}$$

is the radial function. f_0 is a normalization constant that is determined by

$$2\pi \int_0^{\pi/2} f(\theta)\theta \, d\theta = 1 \tag{11}$$

in the small angle approximation. Using

$$\int x\sqrt{a - x^2}\, dx = -\frac{1}{3}(a - x^2)^{3/2} \tag{12}$$

we obtain

$$\frac{1}{f_0} = -\frac{2\pi}{3}\left[(\theta_{\rm out}^2 - \theta_{\rm in}^2)^{3/2} - (\theta_{\rm in}^2 - \theta_{\rm in}^2)^{3/2} - (\theta_{\rm out}^2)^{3/2} + (\theta_{\rm in}^2)^{3/2} + (\theta_{\rm out}^2 - \theta_{\rm out}^2)^{3/2} - (\theta_{\rm out}^2 - \theta_{\rm in}^2)^{3/2}\right] \tag{13}$$

$$= -\frac{2\pi}{3}\left[(\theta_{\rm out}^2 - \theta_{\rm in}^2)^{3/2} - (\theta_{\rm out}^2)^{3/2} + (\theta_{\rm in}^2)^{3/2} - (\theta_{\rm out}^2 - \theta_{\rm in}^2)^{3/2}\right] \tag{14}$$

$$= -\frac{2\pi}{3}\left[-(\theta_{\rm out}^2)^{3/2} + (\theta_{\rm in}^2)^{3/2}\right] \tag{15}$$

$$= \frac{2\pi}{3}\left[\theta_{\rm out}^3 - \theta_{\rm in}^3\right] \tag{16}$$

3.1.2 Spherical formulation

The shell function on a sphere is given by

$$f(\theta) = f_0 \begin{cases} \sqrt{\sin^2\theta_{\rm out} - \sin^2\theta} - \sqrt{\sin^2\theta_{\rm in} - \sin^2\theta} & \text{if } \theta \leq \theta_{\rm in} \\ \sqrt{\sin^2\theta_{\rm out} - \sin^2\theta} & \text{if } \theta_{\rm in} < \theta \leq \theta_{\rm out} \\ 0 & \text{if } \theta > \theta_{\rm out} \end{cases} \quad (17)$$

The normalization constant f_0 is determined by

$$2\pi \int_0^{\pi/2} f(\theta) \sin\theta \, {\rm d}\theta = 1 \quad (18)$$

Using

$$\int \sin x \sqrt{a - \sin^2 x} \, {\rm d}x = -\frac{\cos x \sqrt{\cos 2x + 2a - 1}}{2\sqrt{2}} - \frac{a-1}{2} \ln\left(\sqrt{2}\cos x + \sqrt{\cos 2x + 2a - 1}\right) \quad (19)$$

and

$$2a = 2\sin^2\theta_0 = 1 - \cos 2\theta_0 \quad (20)$$

we can write

$$\int \sin x \sqrt{\sin^2\theta_0 - \sin^2 x} \, {\rm d}x = -\frac{\cos x \sqrt{\cos 2x - \cos 2\theta_0}}{2\sqrt{2}} + \frac{\cos 2\theta_0 + 1}{4} \ln\left(\sqrt{2}\cos x + \sqrt{\cos 2x - \cos 2\theta_0}\right) \quad (21)$$

For the special case of $x = 0$, the integral becomes

$$-\frac{\sqrt{1 - \cos 2\theta_0}}{2\sqrt{2}} + \frac{\cos 2\theta_0 + 1}{4} \ln\left(\sqrt{2} + \sqrt{1 - \cos 2\theta_0}\right) \quad (22)$$

while for the special case $x = \theta_0$, it reduces to

$$\frac{\cos 2\theta_0 + 1}{4} \ln\left(\sqrt{2}\cos\theta_0\right) \quad (23)$$

Using these equations, we can compute the required integrals:

$$\begin{aligned} I_1 = \int_0^{\theta_{\rm in}} \sin\theta \sqrt{\sin^2\theta_{\rm out} - \sin^2\theta} \, {\rm d}\theta &= \frac{\sqrt{1 - \cos 2\theta_{\rm out}} - \cos\theta_{\rm in}\sqrt{\cos 2\theta_{\rm in} - \cos 2\theta_{\rm out}}}{2\sqrt{2}} \\ &+ \frac{\cos 2\theta_{\rm out} + 1}{4} \ln\left(\frac{\sqrt{2}\cos\theta_{\rm in} + \sqrt{\cos 2\theta_{\rm in} - \cos 2\theta_{\rm out}}}{\sqrt{2} + \sqrt{1 - \cos 2\theta_{\rm out}}}\right) \end{aligned} \quad (24)$$

$$I_2 = \int_0^{\theta_{\rm in}} \sin\theta \sqrt{\sin^2\theta_{\rm in} - \sin^2\theta} \, {\rm d}\theta = \frac{\sqrt{1 - \cos 2\theta_{\rm in}}}{2\sqrt{2}} + \frac{\cos 2\theta_{\rm in} + 1}{4} \ln\left(\frac{\sqrt{2}\cos\theta_{\rm in}}{\sqrt{2} + \sqrt{1 - \cos 2\theta_{\rm in}}}\right) \quad (25)$$

$$\begin{aligned} I_3 = \int_{\theta_{\rm in}}^{\theta_{\rm out}} \sin\theta \sqrt{\sin^2\theta_{\rm out} - \sin^2\theta} \, {\rm d}\theta &= \frac{\cos\theta_{\rm in}\sqrt{\cos 2\theta_{\rm in} - \cos 2\theta_{\rm out}}}{2\sqrt{2}} \\ &+ \frac{\cos 2\theta_{\rm out} + 1}{4} \ln\left(\frac{\sqrt{2}\cos\theta_{\rm out}}{\sqrt{2}\cos\theta_{\rm in} + \sqrt{\cos 2\theta_{\rm in} - \cos 2\theta_{\rm out}}}\right) \end{aligned} \quad (26)$$

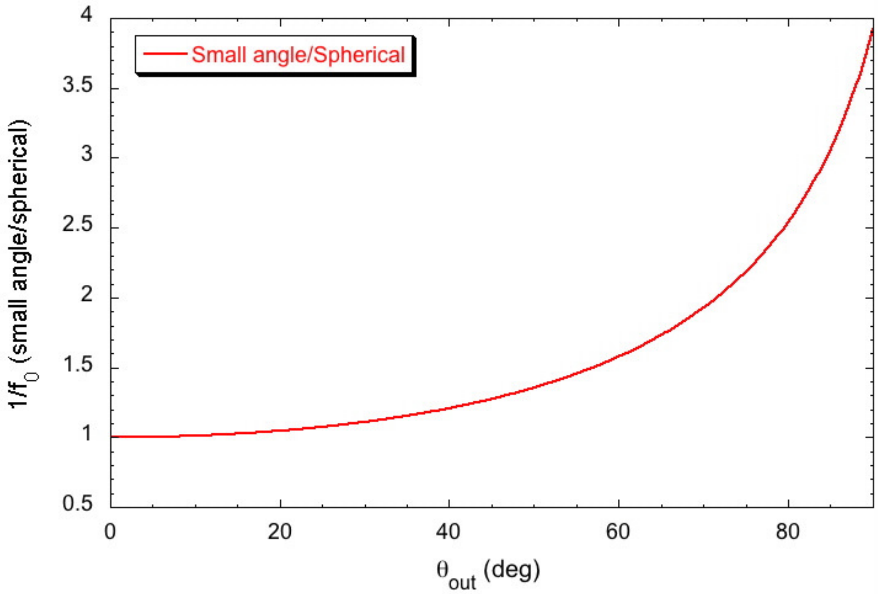

Figure 1: Ratio in $\frac{1}{f_0}$ between the small angle approximation and the correct spherical computation as function of $\theta_{\rm out}$ for $\theta_{\rm in} = \frac{2}{3}\theta_{\rm out}$.

We thus obtain

$$\begin{aligned} \frac{1}{2\pi f_0} &= I_1 - I_2 + I_3 \\ &= \frac{\sqrt{1-\cos 2\theta_{\rm out}} - \sqrt{1-\cos 2\theta_{\rm in}}}{2\sqrt{2}} \\ &+ \frac{\cos 2\theta_{\rm out}+1}{4} \ln\left(\frac{\sqrt{2}\cos\theta_{\rm out}}{\sqrt{2}+\sqrt{1-\cos 2\theta_{\rm out}}}\right) - \frac{\cos 2\theta_{\rm in}+1}{4} \ln\left(\frac{\sqrt{2}\cos\theta_{\rm in}}{\sqrt{2}+\sqrt{1-\cos 2\theta_{\rm in}}}\right) \end{aligned} \quad (27)$$

The ratio in $\frac{1}{f_0}$ between the small angle approximation and the correct spherical computation is shown in Fig. 1 as function of $\theta_{\rm out}$ for $\theta_{\rm in} = \frac{2}{3}\theta_{\rm out}$.

4 Spectral models

4.1 PowerLaw

The power law spectral model is defined by

$$I(E) = k \left(\frac{E}{p}\right)^{\gamma} \quad (28)$$

where

- k is the normalization of the power law (units: ph cm^{-2} s^{-1} MeV^{-1}),
- p is the pivot energy (units: MeV), and
- γ is the spectral index (which is usually negative).

Each of the 3 parameters is factorised into a scaling factor and a value, i.e. $k = k_s k_v$, $p = p_s p_v$, and $\gamma = \gamma_s \gamma_v$. The `GModelSpectralPlaw::eval_gradients` returns the gradients with respect to the parameter value of the factorisation. Note that for any parameter $a = a_s a_v$:

$$\frac{\delta I}{\delta a_v} = \frac{\delta I}{\delta a}\frac{\delta a}{\delta a_v} = \frac{\delta I}{\delta a} a_s \tag{29}$$

The parameter value gradients for the power law are given by

$$\frac{\delta I}{\delta k_v} = k_s \left(\frac{E}{p}\right)^{\gamma} = \frac{I(E)}{k_v} \tag{30}$$

$$\frac{\delta I}{\delta p_v} = -\frac{\gamma}{p_v} k \left(\frac{E}{p}\right)^{\gamma} = -\frac{\gamma}{p_v} I(E) \tag{31}$$

$$\frac{\delta I}{\delta \gamma_v} = \gamma_s \ln\left(\frac{E}{p}\right) k \left(\frac{E}{p}\right)^{\gamma} = \gamma_s \ln\left(\frac{E}{p}\right) I(E) \tag{32}$$

4.2 PowerLaw2

This flavour of the power law spectral model uses the integral flux f within the energy range $E_{\rm min}$ and $E_{\rm max}$ as free parameter instead of the normalization k. The integral flux f is given by

$$f = \int_{E_{\rm min}}^{E_{\rm max}} I(E) {\rm d}E \tag{33}$$

The power law model is then defined by

$$I(E) = \tilde{k}\, E^{\gamma} \tag{34}$$

where

$$\tilde{k} = \begin{cases} \dfrac{f}{\ln E_{\rm max} - \ln E_{\rm min}} & \text{if } \gamma = -1 \\[2ex] \dfrac{f(1+\gamma)}{E_{\rm max}^{\gamma+1} - E_{\rm min}^{\gamma+1}} & \text{else} \end{cases} \tag{35}$$

is obtained from Eq. (33).

Each of the 4 parameters is factorised into a scaling factor and a value, e.g. $f = f_s f_v$ and $\gamma = \gamma_s \gamma_v$. It is assumed that $E_{\rm min}$ and $E_{\rm max}$ are fixed parameters, and `GModelSpectralPlaw2::eval_gradients` returns valid gradients only for f_v and γ_v.

The flux value gradient f_v is given by

$$\frac{\delta I}{\delta f_v} = \frac{\delta I}{\delta f}\frac{\delta f}{\delta f_v} = \frac{\delta I}{\delta f} f_s = \frac{\delta \tilde{k}}{\delta f} E^{\gamma} f_s = \frac{\tilde{k}}{f} E^{\gamma} f_s = \frac{I(E)}{f} f_s = \frac{I(E)}{f_v} \tag{36}$$

The index value gradient γ_v is given by

$$\frac{\delta I}{\delta \gamma_v} = \frac{\delta I}{\delta \gamma}\frac{\delta \gamma}{\delta \gamma_v} = \frac{\delta I}{\delta \gamma}\gamma_s = \left(\frac{\delta \tilde{k}}{\delta \gamma} E^{\gamma} + \tilde{k}\, E^{\gamma} \ln E\right)\gamma_s = \left(\frac{1}{\tilde{k}}\frac{\delta \tilde{k}}{\delta \gamma} + \ln E\right)\tilde{k} E^{\gamma}\gamma_s = \left(\frac{1}{\tilde{k}}\frac{\delta \tilde{k}}{\delta \gamma} + \ln E\right) I(E)\gamma_s \tag{37}$$

where

$$\frac{\delta \tilde{k}}{\delta \gamma} = \begin{cases} 0 & \text{if } \gamma = -1 \\[2ex] \dfrac{f\left(E_{\rm max}^{\gamma+1} - E_{\rm min}^{\gamma+1}\right) - f(1+\gamma)\left(E_{\rm max}^{\gamma+1}\ln E_{\rm max} - E_{\rm min}^{\gamma+1}\ln E_{\rm min}\right)}{\left(E_{\rm max}^{\gamma+1} - E_{\rm min}^{\gamma+1}\right)^2} & \text{else} \end{cases} \tag{38}$$

Note that for $\gamma \neq -1$

$$\frac{1}{\tilde{k}} \frac{\delta \tilde{k}}{\delta \gamma} = \frac{1}{1+\gamma} - \frac{\left(E_{\rm max}^{\gamma+1} \ln E_{\rm max} - E_{\rm min}^{\gamma+1} \ln E_{\rm min} \right)}{\left(E_{\rm max}^{\gamma+1} - E_{\rm min}^{\gamma+1} \right)} \tag{39}$$