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Pulsar timing

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List of Abbreviations and Symbols

$\odot$	symbol that indicates the Sun
$\oplus$	symbol that indicates the Earth
$A_{\mu\nu}$	tensor with indices which assume the values $0, \ldots, 3$
a	semi-major axis of the binary system
c	speed of light
e	eccentricity
g	metric tensor's determinant
G	Gravitational constant
τ	proper time of the pulsar
P_b	pulsar orbital period
u	eccentric anomaly
$\dot{x}$	derivative with respect to the proper time
∇_μ	covariant derivative
$\Box$	d'Alembertian
$f'(R)$	differentiation with respect to the argument
$\{^{\lambda}_{\mu\nu}\}$	Levi-Civita connection
$(\mu\nu)$	symmetrization of indices
$[\mu\nu]$	anti-symmetrization of indices

Introduction

Pulsars are remarkably precise "celestial clocks" that can be used to explore many different aspects of physics and astrophysics. Most applications of pulsars involve a technique called "pulsar timing", which is the measurement of the time of arrival of photons emitted by the pulsar. The amount of information one can get heavily depends on the measurement's precision of the pulse arrival time, which scales as the pulse width divided by the signal to noise ratio. The most recent example is the 5.75-ms pulsar J0437-4517, which yields a mean square residual between model and observations of 100ns over an observing time of one year or longer [1]. It can be seen that pulsar timing precision can be compared to that of the atomic clocks, even though pulsars are obviously less stable. The problem of stabilty can be partially solved if we look at pulsars with smaller period derivatives, since observations show that timing stability for this class of pulsar increases. If millisecond pulsars perform better than terrestrial clocks, they could be used to define a standard time by timing the most stable millisecond pulsars against each other. However there are problems, mainly poorly understood mechanisms that act on the pulsars, and there is also the fact that, while the current definition of second is based on a physical phenomenon which can be reproduced in a lab, a definition of second based on pulsar timing would not have this feature. Nevertheless pulsars and pulsar timing can be used for a wide array of studies, like testing theories of gravity, studying the magnetic field of the Galaxy and the interior of neutron stars and so on. The aim of this thesis is to analyze the time delay effects on pulsar timing for binary systems using a Yukawa gravitational potential in $f(\mathrm{R})$ gravity. In the first chapter we will introduce the time delay effects and we will calculate them both for classical and general reltivstic theory. In the second chapter we will present the basis of $f(\mathrm{R})$ theories and we will show a $f(\mathrm{R})$ theory with Yukawa potential. Finally, in the third chapter, we will work out the expression for the time delays for binary systems in a Yukawa $f(\mathrm{R})$ theory and we will compare them to general relavistic results.

Chapter 1

Time delay for binary systems

Let us consider a signal emitted by a pulsar, which is, basically, an electromagnetic wave. Now, the time of arrival of the signal on Earth will be influenced by three major factors:

1. The Earth orbits around the Sun and there is a difference of arrival time between the case in which the Earth is "on the same side" of the pulsar and the one in which is on the other side of its own orbit around the sun;
2. There are general relativistic effects given by the gravitational field of the solar system;
3. The first two effects cause a shift in the coordinate time, but a clock placed in a laboratory on Earth measures its own proper time.

These delays are known respectively as Roemer, Shapiro and Einstein delay [2]. These time delays are not exclusive to the Earth-Sun system, but they are also present in a pulsar-pulsar system, with the difference that a binary pulsar is a fairly relativistc system and its description must make use of a full general relativistic formalism.

1.1 Solar System time delay

1.1.1 Roemer time delay

It is known that light takes approximately 500s to get from the Sun to the Earth, so the position of Earth on its orbit around the sun is a big factor in the modulation of arrival times. This can be easily understood by looking at Figure 1.1: depending on the position of the Earth on its orbit around the Sun, the signal coming from the pulsar will take more time to arrive if the Earth is on the left side of the Sun and

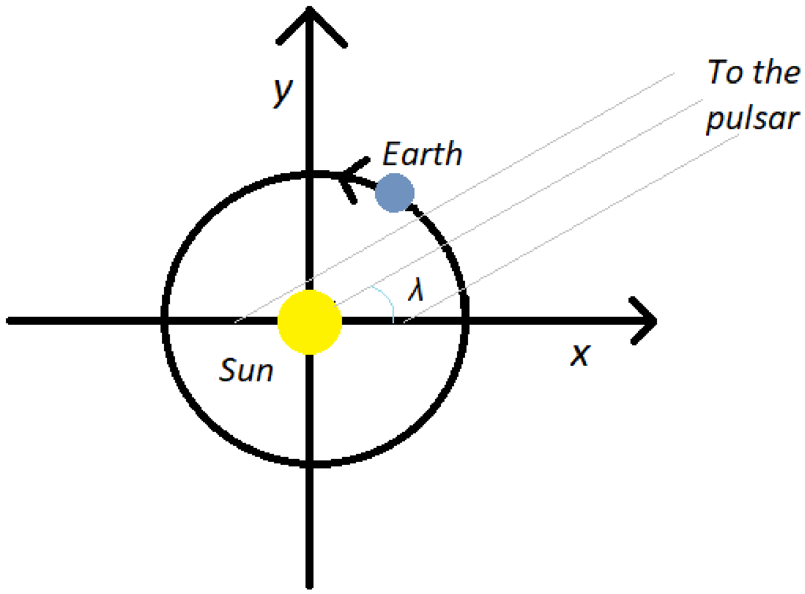

Figure 1.1: The plane (x,y) represents the plane of orbit of the Earth around the Sun. The angle λ is the ecliptic longitude of the pulsar.

it will take less time to arrive if it's on the right side of the Sun. Let us assume, for simplicity, that Earth travels on a circular orbit around the sun and its angular velocity is Ω. Then, if we call t_0 the time that the signal takes from the Sun to the Earth, we will find that the modulation at ecliptic longitude λ will be given by

$$\Delta_{R,\odot} = t_0 \cos(\Omega t - \lambda), \tag{1.1}$$

where the subscript R stands for Roemer, while $\odot$ indicates that this is the Roemer delay with respect to the Sun and that the correction is caused by the motion of the observer and not by the motion of the source.
This is, obviously, correct for a pulsar that lies in the ecliptic plane, while, for a pulsar that also has an ecliptic latitude, the formula becomes

$$\Delta_{R,\odot} = t_0 \cos(\Omega t - \lambda) \cos\beta, \tag{1.2}$$

where β represent the angle of ecliptic latitude. From the formula we can see that the modulation vanishes for pulsars in the direction of the poles of the ecliptic and it has its maximum amplitude when the pulsar is in the ecliptic plane.
If we make a variation in the angles $\delta\lambda, \delta\beta$, the resulting variation in $\Delta_{R,\odot}$ will be

$$\delta(\Delta_{R,\odot}) = t_0 \delta\lambda \sin(\Omega t - \lambda) \cos\beta - t_0 \delta\beta \cos(\Omega t - \lambda) \sin\beta. \tag{1.3}$$

If we have a resolution of $\delta t = 0,2$ ms on the arrival time of the pulse and we choose $\beta \simeq 45°$ we will have an accuracy on the angles of the order

$$\delta\lambda \simeq \delta\beta \simeq \sqrt{2}\frac{\delta t}{t_0} \simeq 0,1 arcsec. \tag{1.4}$$

Nevertheless, we will need a better precision for pulsar timing so we have to refine our approximations. First of all, we can not assume a circular orbit for Earth, so we have to use an elliptic one. Secondly, we will also have to take into account the rotation of Earth arounf its axis. Lastly we have to consider the motion of the Sun around the Solar System Barycenter. To take all of this into account we can simply refer the time of arrival to the Solar System Barycenter (SSB). In order to do so we will have to define the following quantities:

1. $\boldsymbol{r_{oe}}$, the vector from the observer to the center of the Earth;
2. $\boldsymbol{r_{es}}$, the vector from the center of the Earth to the center of the Sun;
3. $\boldsymbol{r_{sb}}$, the vector from the center of the Sun to the SSB.

Once we have done that, we will have that the distance from the observer to the SSB will be

$$\boldsymbol{r_{ob}} = \boldsymbol{r_{oe}} + \boldsymbol{r_{es}} + \boldsymbol{r_{sb}}. \tag{1.5}$$

This implies that, to obtain the baricentric time of arrival, we have to add to the time we measured in the laboratory the following quantity

$$\Delta_{R,\odot} = -\boldsymbol{r_{ob}} \cdot \frac{\hat{\boldsymbol{n}}}{c}, \tag{1.6}$$

where $\hat{\boldsymbol{n}}$ is the unit vector from the SSB to the pulsar. We can measure $\boldsymbol{r_{oe}}, \boldsymbol{r_{es}}$ and $\boldsymbol{r_{sb}}$ with good accuracy, so we can get $\hat{\boldsymbol{n}}$ from a measure of $\Delta_{R,\odot}$.

1.1.2 Shapiro time delay

The above computation neglects the deviation of light caused by the gravitational field of the Solar System. In order to consider that, let's start by writing the space-time interval generated by a weak and static Newtonian source to linear order in the metric perturbation ϕ

$$ds^2 = -[1 + 2\phi(\boldsymbol{x})]c^2dt^2 + [1 - 2\phi(\boldsymbol{x})]d\boldsymbol{x}^2. \tag{1.7}$$

Since we have that for the Solar System $|\phi(\boldsymbol{x})| \simeq 10^{-6}$ the weak field approximation works very well. Now, knowing that photons travel on light-like geodesics $ds^2 = 0$, to the lowest order in ϕ

$$cdt = \pm[1 - 2\phi(\boldsymbol{x})]|dx| \tag{1.8}$$

Let us call $\boldsymbol{r_p}$ the distance to the fixed pulsar and $\boldsymbol{r_{obs}}$ the position of the observer at the arrival time t_{obs}. Then we will have that the difference in coordinate time between the time of arrival at the observer t_{obs} and the time of emission of the signal by the pulsar t_{em} will be given by

$$c(t_{obs} - t_{em}) = \int_{r_{obs}}^{r_p} |d\boldsymbol{x}|[1 - 2\phi(\boldsymbol{x})] = |\boldsymbol{r_p} - \boldsymbol{r_{obs}}| - 2\int_{r_{obs}}^{r_p} |d\boldsymbol{x}|\phi(\boldsymbol{x}). \quad (1.9)$$

If we introduce the position of the SSB as $\boldsymbol{r_b}$ and recall that $\boldsymbol{n}$ is the unit vector from to the SSB to the pulsar ($\hat{\boldsymbol{n}} = (\boldsymbol{r_b} - \boldsymbol{r_{obs}})/|\boldsymbol{r_b} - \boldsymbol{r_{obs}}|$), then we can write the first piece of the equation above as

$$|\boldsymbol{r_p} - \boldsymbol{r_{obs}}| = |\boldsymbol{r_p} - \boldsymbol{r_b} + \boldsymbol{r_b} - \boldsymbol{r_{obs}}| \simeq |\boldsymbol{r_p} - \boldsymbol{r_b}| + (\boldsymbol{r_b} - \boldsymbol{r_{obs}}) \cdot \hat{\boldsymbol{n}}, \quad (1.10)$$

where the second equality is valid because $|\boldsymbol{r_p} - \boldsymbol{r_b}| \gg |\boldsymbol{r_b} - \boldsymbol{r_{obs}}|$. Then, if we define $\boldsymbol{r_b} - \boldsymbol{r_o bs} \equiv \boldsymbol{r_{ob}}$ we will have

$$t_{obs} \simeq (t_{em} + \frac{1}{c}|\boldsymbol{r_p} - \boldsymbol{r_b}|) + \frac{1}{c}\boldsymbol{r_{ob}} \cdot \hat{\boldsymbol{n}} - \frac{2}{c}\int_{r_{obs}}^{r_p} |d\boldsymbol{x}|\phi(\boldsymbol{x}). \quad (1.11)$$

The first term is the time of arrival at the SSB, which is the time of arrival of the pulse at the Solar System Barycenter if there were no gravitational effects. So we can write

$$t_{SSB} = t_{obs} - \frac{1}{c}\boldsymbol{r_{ob}} \cdot \hat{\boldsymbol{n}} + \frac{2}{c}\int_{r_{obs}}^{r_p} |d\boldsymbol{x}|\phi(\boldsymbol{x}). \quad (1.12)$$

As we can see from 1.6, the second addendum is none other that the Roemer delay; the third chunk is what is called Shapiro delay and it represents the effects of the gravitational field of the solar system on the path taken by light. To be precise the Shapiro delay is defined as

$$\Delta_{S,\odot} = -\frac{2}{c}\int_{r_{obs}}^{r_p} |d\boldsymbol{x}|\phi(\boldsymbol{x}), \quad (1.13)$$

so the last term in 1.12 is minus the Shapiro delay. So, putting everything together, we get

$$t_{SSB} = t_{obs} + \Delta_{R,\odot} - \Delta_{S,\odot}. \quad (1.14)$$

Our main focus lies on how to compute the Shapiro delay. To do that, we will consider Figure 1.2 as a reference. Consider a photon emitted by the pulsar, which reaches the observer on Earth when the pulsar-Earth-Sun angle has the value θ. Let us, then, call P a generic point on the straight trajectory of the photon and define ρ as the distance between P and the Earth and r as the distance between P and the Sun. Let us call $r_{es} = 1au$ the distance between the Eath and the Sun and, as can be seen from the picture, the following holds true:

$$r^2 = (r_{es} + \rho\cos\theta)^2 + (\rho\sin\theta)^2, \tag{1.15}$$

which means, defining $u = \rho/r_{es}$,

$$r = r_{es}\sqrt{(u^2 + 1 + 2u\cos\theta)}. \tag{1.16}$$

Now, remembering the expression of the Newtonian potential $\phi = -(GM_{\odot}/r)$

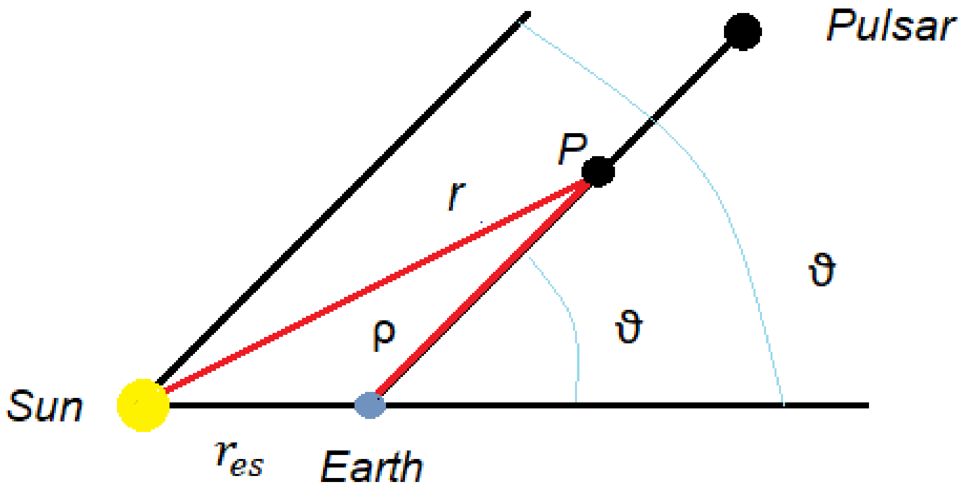

Figure 1.2: A visual representation of the configuration used to compute the Shapiro delay for the Solar System.

we will have

$$\Delta_{S,\odot} = \frac{2GM_\odot}{c^3}\int_0^d \frac{d\rho}{r} = \frac{2GM_\odot}{c^3}\int_0^{\bar{u}} \frac{du}{\sqrt{(u^2+1+2u\cos\theta)}}, \quad (1.17)$$

where $\bar{u} = d/r_{es}$ and d is the distance between Earth and the pulsar. This integral can be pretty difficult to solve and, in order to do that, we will use a trick. Let us add and subtract the delay at a given angle, for example (and to keep things simple) when $\cos\theta = 0$. Doing that will result in

$$\Delta_{S,\odot} = \frac{2GM_\odot}{c^3}\int_0^{\bar{u}} \frac{du}{\sqrt{(u^2+1)}} + \frac{2GM_\odot}{c^3}\int_0^{\bar{u}} du\left[\frac{1}{\sqrt{(u^2+1+2u\cos\theta)}} - \frac{1}{\sqrt{(u^2+1)}}\right]. \quad (1.18)$$

Once we've done that we can notice that the first term is a fixed quantity that grows logarithmically for d/r_{es}

$$\frac{2GM_\odot}{c^3}\int_0^{\bar{u}} \frac{du}{\sqrt{(u^2+1)}} = \frac{2GM_\odot}{c^3}\operatorname{arcsinh}(\bar{u}) \simeq \frac{2GM_\odot}{c^3}\log\left(\frac{2d}{r_{es}}\right). \quad (1.19)$$

As we can see, this is a constant shift that has to be added to the time that the photon takes from the pulsar to the SSB. The only thing that remains to compute is the second line of 1.18. This integral can also be evaluated it in the limit $\bar{u} \to \infty$, so that we get

$$\int_0^{\infty} du\left[\frac{1}{\sqrt{(u^2+1+2u\cos\theta)}} - \frac{1}{\sqrt{(u^2+1)}}\right] = -\log(1+\cos\theta). \quad (1.20)$$

Getting the two pieces together we have

$$\Delta_{S,\odot} = \frac{2GM_\odot}{c^3}\log\left(\frac{2d}{r_{es}}\right) - \frac{2GM_\odot}{c^3}\log(1+\cos\theta). \quad (1.21)$$

A plot of the function $-\log(1+\cos\theta)$ is shown in Figure 1.3. From that we can see that there is a divergence in $\theta = \pi/2$ which is the angle that corresponds to a photon passing through the center of the Sun before getting to Earth. Of course, this is just a ficticious divergence for two reasons: first, a signal would not get through the Sun but would simply be absorbed; secondly, the Newtonian potential $-GM/r$ is valid only outside the Sun and this means that this expression is valid for signals that, at most, graze the Sun.
The equation for the Shapiro delay can also be rewritten as

$$\Delta_{S,\odot} = \frac{2GM_\odot}{c^3}\left[\log\left(\frac{d}{r_{es}}\right) - \log\left(\frac{1+\cos\theta}{2}\right)\right], \quad (1.22)$$

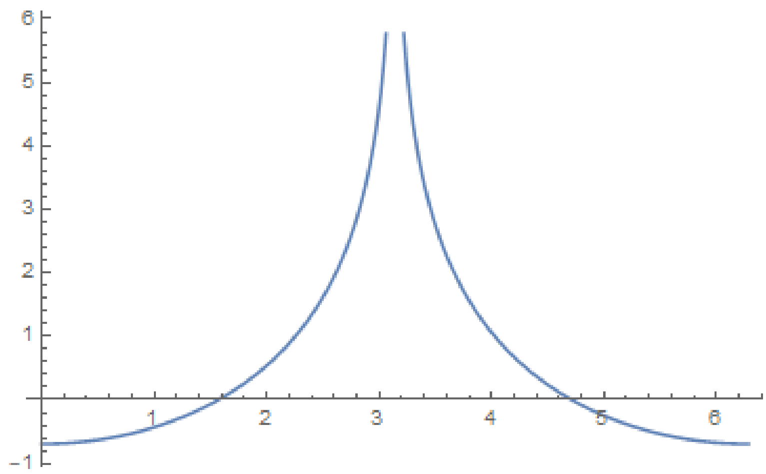

Figure 1.3: Plot showing $-\log(1+\cos\theta)$ as a function of θ.

in order to better stress out that $\Delta_{S,\odot}$ is an always positive quantity, which is what we expect if we reverse 1.14 to obtain t_{obs}, since the effect of the gravitational field on the signal must be a delay in the time of arrival to the observer.

1.1.3 Einstein time delay

The delays computed until now represent a shift in the coordinate time, but this is not the same as the time measured by a clock in a laboratory. A clock in a laboratory located at position $\boldsymbol{x}_{obs}$ will measure its proper time τ. To take this into account, we can use the fact that proper time is related to coordinate time by

$$c^2 d\tau^2 = -[1+2\phi(\boldsymbol{x}_{obs})]c^2 dt^2 + [1-2\phi(\boldsymbol{x}_{obs})]d\boldsymbol{x}_{obs}^2. \tag{1.23}$$

This implies that, to the first order in the parameters $\phi(\boldsymbol{x}_{obs})$ and v_{obs}, we will have

$$\frac{d\tau}{dt} \simeq 1 + \phi(\boldsymbol{x}_{obs}) - \frac{v_{obs}^2}{2c^2}. \tag{1.24}$$

The term $v_{obs}^2/2c^2$ will give the transverse doppler shift, while $\phi(\boldsymbol{x}_{obs})$ will give the gravitational redshift. We can now integrate to obtain

$$\tau \simeq t + \int^t dt' \left[\phi(\boldsymbol{x}_{obs}) - \frac{v_{obs}^2}{2c^2}\right], \tag{1.25}$$

where the lower limit of integration is left blank since it correspond to an arbitrary constant shift in the origin of τ. We can then rewrite the above expression as

$$t \simeq \tau + \Delta_{E,\odot}, \tag{1.26}$$

where

$$\Delta_{E,\odot} = \int^t dt' \left[\phi(\boldsymbol{x}_{obs}) - \frac{v_{obs}^2}{2c^2} \right]. \tag{1.27}$$

This is called Einstein delay. In order to compute v_{obs} we should take into account both the velocity of motion of the Earth around the Sun $v_\oplus$ and the velocity of rotation of Earth around its axis, but, since the latter represents only a small correction, we will assume $v_{obs} \simeq v_\oplus$. We take the Earth in an elliptic orbit around the Sun with semi-major axis a. We then know that, in a Keplerian orbit, the relation for the total kinetic plus potential energy of the system can be written as a function of the semi-major axis in the following way

$$E = -\frac{GM\mu}{2a}, \tag{1.28}$$

where μ is the reduced mass of the Earth-Sun system (which basically corresponds to Earth's mass) and M in the total mass (which can be practically identified with the mass of the Sun). Since, on the other hand,

$$E = \frac{1}{2}\mu v_\oplus^2 - \frac{GM\mu}{r}, \tag{1.29}$$

we will have

$$\frac{1}{2} v_\oplus^2 = \frac{GM_\odot}{r} - \frac{GM_\odot}{2a}. \tag{1.30}$$

So this means that

$$\frac{d\Delta_{E,\odot}}{dt} \simeq \frac{v_\oplus^2}{2c^2} - \phi = \frac{2GM_\odot}{c^2}\left(\frac{1}{r} - \frac{1}{4a}\right). \tag{1.31}$$

A constant part in this expression is incorporated in the definition of atomic time, which is defined as the time measured by an atomic clock at a fixed distance a from the Sun. The dependence on r however introduces a modulation, due to the ellepticity of Earth's orbit.

1.2 Time delay for binary pulsars

For a pulsar in a binary system we can proceed exactly as we did for the Earth-Sun system which, in this case, will be by making a transformation from the pulsar proper time to the coordinate time fo the pulsar-companion baricenter. We will then have a Roemer, Shapiro and Einstein delays associated with the pulsar-companion system. The main difference with the Solar System effects is that, in the case of a binary pulsars, general relativistic effects are much more important, since a binary pulsar system is a relativistic system. This implies that the calculations will become more complex, since we will have to treat a fully general relativistic two-body problem.

1.2.1 Reomer time delay

Referring the emission time to the barycenter of the pulsar-companion system, we will have Roemer and Shapiro delays, as we have found for the Solar System. If we refer again to Figure 1.1, we will have that the Roemer delay will be given by

$$\Delta_R = \hat{\boldsymbol{z}} \cdot \frac{\boldsymbol{x}_1(t)}{c}, \tag{1.32}$$

where $\boldsymbol{x}_1$ is the distance between the pulsar and the center of mass of the system. This means that we need to find the explicit form of the orbit $\boldsymbol{x}_1(t)$. Let's start by considering a Keplerian orbit and neglecting general relativistic corrections. By using polar coordinates in the plane of the orbit (r_1, ψ), we will have that the parametric form of the Keplerian equations of motions, as a function of the eccentric anomaly u, will be

$$r_1(u) = a_1(1 - e\cos u), \tag{1.33}$$

$$\cos\psi(u) = \frac{\cos u - e}{1 - e\cos u}, \tag{1.34}$$

$$\sin\psi(u) = \sqrt{(1-e^2)}\frac{\sin u}{1 - e\cos u}, \tag{1.35}$$

where a_1 is the semi-major axis of the orbit and we have introduced $\sin\psi(u)$ because we will need it for the following computations.
It can be easily seen that the minimum value for r_1 is obtained when $u = 0$, which also implies that $\psi = 0$. All of this means that the angle ψ can be measured from the periastron, and that the measure of the angle from the node line will be $\omega + \psi(u)$. Looking again at Figure 1.1 we have that the Roemer delay will be

$$\Delta_R = r_1(u)\sin i \sin(\omega + \psi(u)). \tag{1.36}$$

We can then expand $\sin(\omega + \psi(u))$ and substitute the expression of $\sin\psi$ and $\cos\psi$ and we will get

$$\begin{aligned}\Delta_R &= \frac{r_1(u)}{1 - e\cos u}\sin i[(\cos u - e)\sin\omega + \sqrt{(1-e^2)}\sin u\cos\omega] \\ &= a_1 \sin i[(\cos u - e)\sin\omega + \sqrt{(1-e^2)}\sin u \cos\omega]. \end{aligned} \tag{1.37}$$

We now have the Keplerian result, but the general relativistic corrections are quite large, so we have to go beyond the Keplerian orbit and include the post-Newtonian corrections to the 1PN order [3]. To obtain the equations of motion we start from the 1PN Lagrangian. This can be written as

$$\mathcal{L}_{1PN} = \mathcal{L}_N + \mathcal{L}_2, \tag{1.38}$$

where L_N is the Newtonian Lagrangian and L_2 represents the 1PN correction. We will focus on the latter, whose expression is

$$\mathcal{L}_2 = \frac{1}{8}m_p v_p^4 + \frac{1}{8}m_c v_c^4 + \frac{Gm_p m_c}{2r}\left[3v_p^2 + 3v_c^2 - 7(v_p v_c) - (Nv_p)(Nv_c) - G\frac{m_p + m_c}{r}\right], \tag{1.39}$$

where (v_p, v_c) are the simultaneous velocities in a given harmonic coordinate system, (m_p, m_c) are the masses of the pulsar and the companion, r is the relative position vector and N is the relative position versor. If we define the variable

$$\boldsymbol{X} = \frac{m_p^* \boldsymbol{x}_p + m_c^* \boldsymbol{x}_c}{m_p^* + m_c^*}, \tag{1.40}$$

where m_i^* is defined as

$$m_i^* = m_i + \frac{m_i v_i^2}{2c^2} - \frac{Gm_p m_c}{2c^2}, \tag{1.41}$$

we will get that the equations of motion will assume the form

$$\frac{d^2\boldsymbol{X}}{dt^2} = 0. \tag{1.42}$$

If we put ourselves in the non realtivistic case, this simply means that the center-of-mass is not accelerated. In the relativistic case, since we have corrections of the order $\mathcal{O}(v^2/c^2)$[1], $\boldsymbol{X}$ can be seen as a "center-of-energy". Invariance under time translations and rotations lead to the conservation of energy and angular momentum. From the latter we also get that the equation for R describes the motion in a plane, just like in the Newtonian case. Let's now introduce the following notation:

1. $m = m_p + m_c$ is the total mass;
2. $\mu = \frac{m_p m_c}{m_p + m_c}$ is the reduced mass ;
3. $\nu = \frac{m_p m_c}{(m_p + m_c)^2}$ is the symmetric mass ratio ;
4. $\epsilon = \frac{E}{\mu}$ is the energy per unit of μ ;
5. $\boldsymbol{j} = \frac{\boldsymbol{J}}{\mu}$ is the angular momentum per unit of μ.

[1]The Post Newtonian expansion is carried out in the small parameter (v^2/c^2), so the first Post Newtonian approximation will contain corrections of the order $\mathcal{O}(v^2/c^2)$ [4].

If we then apply Noether Theorem[2] to the Lagrangian 1.39 we can get the expressions of the conserved quantities

$$\epsilon = \frac{1}{2}v^2 - \frac{Gm}{r} + \frac{3}{8}(1-3\nu)\frac{v^4}{c^2} + \frac{Gm}{2rc^2}\left[(3+\nu)v^2 + \nu(\hat{\boldsymbol{r}}\cdot\boldsymbol{v})^2 - \frac{Gm}{r}\right], \quad (1.43)$$

and

$$\boldsymbol{j} = \left[1 + \frac{1}{2}(1-3\nu)\frac{v^2}{c^2} + (3+\nu)\frac{Gm}{r}\right]\boldsymbol{r}\times\boldsymbol{v}. \quad (1.44)$$

Using planar polar coordinates (r, ψ) the first integrals of the equations of motion give

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = A + \frac{2B}{r} + \frac{C}{r^2} + \frac{D}{r^3}, \quad (1.45)$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{H}{r^2} + \frac{I}{r^3}, \quad (1.46)$$

where the factors A,B...,I are the following polynomials in ϵ and $\boldsymbol{j}$

$$A = 2\epsilon\left[1 + \frac{3}{2}(3\nu-1)\frac{\epsilon}{c^2}\right], \quad (1.47)$$

$$B = Gm\left[1 + (7\nu-6)\frac{\epsilon}{c^2}\right], \quad (1.48)$$

$$C = -j^2\left[1 + 2(3\nu-1)\frac{\epsilon}{c^2}\right] + (5\nu-10)\frac{G^2m^2}{c^2}, \quad (1.49)$$

$$D = (8-3\nu)\frac{Gmj^2}{c^2}, \quad (1.50)$$

$$H = j\left[1 + (3\nu-1)\frac{\epsilon}{c^2}\right], \quad (1.51)$$

$$I = (2\nu-4)\frac{Gmj}{c^2}. \quad (1.52)$$

We have that in the limit $c \to \infty$ the D term goes to zero, while the others reduce to their Newtonian equivalent. Comparing the general relativistic case to the Newtonian one we can see that in the latter the radial equation contains terms up to the order $1/r^2$ while for the former the order gets to $1/r^3$, which could make the integration difficult. To remedy this we can introduce the change of variable first proposed by Damour and Deruelle [3]

$$\bar{r} = r + \frac{D}{2j^2}. \quad (1.53)$$

[2]Noether's theorem states that every differentiable symmetry of the action of a physical system has a corresponding conservation law [5]

Having done that, we can use it in 1.45 in order to get

$$\left(\frac{d\bar{r}}{dt}\right)^2 = a + \frac{2B}{\bar{r}} + \frac{\bar{C}}{\bar{r}^2} + \mathcal{O}\left(\frac{v^4}{c^4}\right), \tag{1.54}$$

with $\bar{C} = C + (BD/j^2)$. Since we will neglect the terms $\mathcal{O}\left(\frac{v^4}{c^4}\right)$, this is practically the Newtonian equation with a different parameter $\bar{r}$. We can apply a similar change of variable in 1.46 to obtain

$$\frac{d\psi}{dt} = \frac{H}{\tilde{r}^2}, \tag{1.55}$$

where the new variable $\tilde{r}$ is defined as

$$\tilde{r} = r - \frac{I}{2H}. \tag{1.56}$$

As a result, the equations can be integrated analitically, obtaining a result similar to the Keplerian one. For 1.45 we get

$$u - e_t \sin u = \frac{2\pi}{P_b} t, \tag{1.57}$$

$$r = a_r(1 - e_r \cos u), \tag{1.58}$$

where the new parameters have the following definitions

$$a_r = -\frac{Gm}{2\epsilon}\left[1 - (\nu - 7)\frac{\epsilon}{2c^2}\right], \tag{1.59}$$

$$e_r^2 = 1 + \frac{2\epsilon}{G^2m^2}\left[1 + (5\nu - 15)\frac{\epsilon}{2c^2}\right]\left[j^2 + (\nu - 6)\frac{G^2m^2}{c^2}\right], \tag{1.60}$$

$$e_t^2 = 1 + \frac{2\epsilon}{G^2m^2}\left[1 + (17 - 7\nu)\frac{\epsilon}{2c^2}\right]\left[j^2 + (2 - 2\nu)\frac{G^2m^2}{c^2}\right], \tag{1.61}$$

$$\frac{2\pi}{P_b} = \frac{(-2\epsilon)^{-2/3}}{Gm}\left[1 - (\nu - 15)\frac{\epsilon}{4c^2}\right]. \tag{1.62}$$

We can see how the eccentricity gets split into a radial eccentricity e_r and a time eccentricity e_t. In the same way, the solution for ψ will be written as a function of an angular eccentricity e_θ

$$\psi = \omega_0 + (1 + k)A_{e_\theta}(u), \tag{1.63}$$

where

$$k = \frac{3Gm}{c^2 a(1-e^2)}, \tag{1.64}$$

$$e_\theta^2 = 1 + \frac{2\epsilon}{G^2m^2}\left[1 + (5\nu - 15)\frac{\epsilon}{2c^2}\right]\left[j^2 - 6\frac{G^2m^2}{c^2}\right], \tag{1.65}$$

$$A_{e_\theta}(u) = 2\arctan\left[\left(\frac{1+e_\theta}{1-e_\theta}\right)^{1/2}\tan\frac{u}{2}\right]. \tag{1.66}$$

As usual, in the limit $c \to \infty$ the three quantities e_r^2, e_t^2 and e_θ^2 reduce to the Newtonian e^2. We therefore have a parametric "quasi-Newtonian" expression for the orbit that we can easily replace in 1.37 in order to obtain the expression for the general relativistic Roemer delay

$$\Delta_R = a_1 \sin i[(\cos u - e_r)\sin\omega + \sqrt{(1-e_\theta^2)}\sin u\cos\omega]. \tag{1.67}$$

We can, additionally, rewrite the expression by defining $e_r = (1+\delta_r)e$ and $e_\theta = (1+\delta_\theta)e$, where δ_r and δ_θ are given by the following expressions

$$\delta_r = \frac{G}{c^2}\frac{3m_p^2 + 6m_pm_c + 2m_c^2}{a(m_p+m_c)}, \tag{1.68}$$

$$\delta_\theta = \frac{G}{c^2}\frac{(7/2)m_p^2 + 6m_pm_c + 2m_c^2}{a(m_p+m_c)}, \tag{1.69}$$

This is useful mainly because we can use these two parameters to test alternative theories of gravity by setting them as free parameters and deriving them by the data.

1.2.2 Einstein time delay

For the Einstein delay we will make the same computation as we did for the Earth-Sun system, since the Newtonian equation of trajectory gives a good enough accuracy. Let's start by defining m_p as the mass of the pulsar and m_c as the mass of the companion. From here onwards the total mass m and the reduced mass μ are to be computed by using the two masses defined above. First, we need to analyze a conceptual point. The signal is emitted by an "hot spot" at a position $\boldsymbol{x}$ on the surface of the pulsar. The Newtonian expression for ϕ in $\boldsymbol{x}$ wil be

$$\phi(\boldsymbol{x}) = -\frac{Gm_p}{c^2|\boldsymbol{x}-\boldsymbol{x_p}|} - \frac{Gm_c}{c^2|\boldsymbol{x}-\boldsymbol{x_c}|}, \tag{1.70}$$

where $\boldsymbol{x_p}$ and $\boldsymbol{x_c}$ are the position of the center of the pulsar and the center of the companion respectively. If we use the data from the Hulse-Taylor binary pulsar we will have that the second term in the expression is of order 10^{-6}, which justifies a weak-field approximation. The self gravity of the pulsar is, however, strong on the surface. For a typical neutron star of radius $r_{NS} \simeq 10km$ and mass $m_p \simeq 1,4M_\odot$ we will have that the first term in 1.70 is around 0,2. However this term is the same all along the trajectory of the pulsar around its companion so it does not introduce a modulation of the time of arrival and can be reabsorbed as a costant rescaling of the proper time T, which is not observable. So we can simply compute the time dependent part of the Einstein delay by using

$$\phi(\boldsymbol{x}) = -\frac{Gm_c}{c^2|\boldsymbol{x} - \boldsymbol{x_c}|}, \tag{1.71}$$

and the weak field approximation. Then, if we look at 1.24 we will have

$$\frac{d\tau}{dt} = 1 - \frac{Gm_c}{c^2|\boldsymbol{x_p} - \boldsymbol{x_c}|} - \frac{v_p^2}{2c^2}, \tag{1.72}$$

where $\boldsymbol{x_p}$ is the pulsar position and v_p is the pulsar velocity. We can find the latter from the following expression

$$v_p = \frac{m_c}{m_p + m_c} v, \tag{1.73}$$

with v the velocity of the center of mass of the system which is given by

$$\frac{1}{2}v^2 - \frac{G(m_p + m_c)}{r} = -\frac{G(m_p + m_c)}{2a}. \tag{1.74}$$

Substituting this expression in 1.72 we get

$$\frac{dT}{dt} = 1 - \frac{G}{c^2}\left[\frac{m_c(m_p + 2m_c)}{m_p + m_c}\frac{1}{r} - \frac{m_c^2}{m_p + m_c}\frac{1}{2a}\right]. \tag{1.75}$$

We can now rewrite everything as a function of the eccentric anomaly u by using the following parametrization of the Keplerian orbit

$$r = a(1 - \cos u), \tag{1.76}$$

$$\cos\psi = \frac{\cos u - e}{1 - e\cos u}, \tag{1.77}$$

and using the fact that the dependence of u on time is given by

$$u - e\sin u = \frac{2\pi}{P_b}(t - t_0), \tag{1.78}$$

where t_0 is a reference time of periastron passage. We can differentiate the last expression to obtain

$$\frac{du}{dt}(1 - e\cos u) = \frac{2\pi}{P_b}, \tag{1.79}$$

an therefore, by playing with the differentials,

$$\frac{dT}{dt} = \frac{du}{dt}\frac{dT}{du} = \frac{2\pi}{P_b}\frac{1}{1 - e\cos u}\frac{\mathrm{d}T}{\mathrm{d}u}. \tag{1.80}$$

We then susbstitute this in 1.75 and use 1.76 and we obtain

$$\begin{aligned}\frac{2\pi}{P_b}\frac{dT}{du} &= \left(1 - \frac{G}{c^2}\frac{2m_c m_p + 3m_c^2}{2a(m_p + m_c)}\right) - e\cos u\left(1 - \frac{G}{c^2}\frac{m_c^2}{2a(m_p + m_c)}\right) \\ &\approx \left(1 - \frac{G}{c^2}\frac{2m_c m_p + 3m_c^2}{2a(m_p + m_c)}\right) \\ &\quad \times \left[1 - e\cos u\left(1 + \frac{G}{c^2}\frac{m_c(m_p + 2m_c)}{a(m_p + m_c)}\right)\right], \end{aligned} \tag{1.81}$$

where in the second line we have discarded terms of the second order or more in G. If we look at the factor in front of the parentheses we can easily see that it's a constant rescaling of the pulsar proper time T. This factor is, then, unobservable since it relates the proper time of the pulsar in the presence of only its own gravitational field to the actual proper time in the presence of the companion's gravitational field and with the added presence of an orbital velocity. This means that we can rescale the pulsar proper time in the following manner

$$T \to \left(1 - \frac{G}{c^2}\frac{2m_c m_p + 3m_c^2}{2a(m_p + m_c)}\right)T. \tag{1.82}$$

We will then have that 1.81 becomes

$$\frac{dT}{du} = \frac{P_b}{2\pi}(1 - e\cos u) - \gamma\cos u, \tag{1.83}$$

where we have defined γ, which is called Einstein parameter, as

$$\gamma = e\frac{P_b}{2\pi}\frac{G}{c^2}\frac{m_c(m_P + 2m_c)}{a(m_p + m_c)} = e\frac{P_b}{2\pi}^{1/3}\frac{G^{2/3}}{c^2}\frac{m_c(m_p + 2m_c)}{(m_p + m_c)^{4/3}}, \tag{1.84}$$

where in the second equality we have just used Kepler's law

$$G\frac{(m_p + m_c)}{a^3} = \frac{2\pi}{P_b}^2, \tag{1.85}$$

to get rid of a. If we write $T = t - \Delta_E$ we will have

$$\frac{2\pi}{P_b}\left(\frac{dt}{du} - \frac{d\Delta_E}{du}\right) = (1 - e\cos u) - \gamma\cos u. \tag{1.86}$$

We can now see, from 1.78, that

$$\frac{2\pi}{P_b}\frac{dt}{du} = 1 - e\cos u, \tag{1.87}$$

so, in the end, 1.76 will become

$$\frac{d\Delta_E}{du} = \gamma\cos u. \tag{1.88}$$

This is very easy to integrate, so we can finally obtain the expression for the Einstein delay in a binary system

$$\Delta_E = \gamma\sin u. \tag{1.89}$$

1.2.3 Shapiro time delay

Let us consider a signal that departs from a pulsar in a binary system and gets to Earth using Figure 1.4 as a reference [6]. In the non-relativistic case the time that

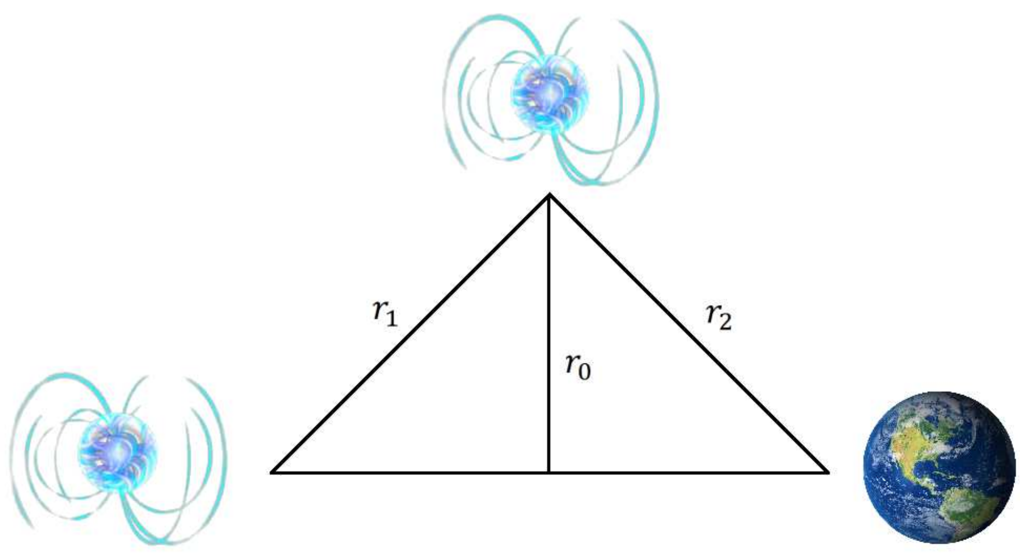

Figure 1.4: A representation of the pulsar-Earth-pulsar system in the equatorial plane

the signal takes from the pulsar to the Earth will simply be

$$c\,t_{NR} = \sqrt{r_1^2 - r_0^2} + \sqrt{r_2^2 - r_0^2}, \tag{1.90}$$

where r_0 is the distance of closest approach of the signal to the companion pulsar. We can now generalize the logic behind 1.90 in order to obtain a relativistic formula, by saying

$$c\,t_R = F(r_1) + F(r_2). \tag{1.91}$$

The goal of our computation will then be to find an expression for $F(r_1)$ and $F(r_2)$. Let's start with the most general case by using a metric with spherical symmetry and signature $(+,-,-,-)$

$$\mathrm{d}s^2 = A(r)c^2\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin\theta^2\mathrm{d}\phi^2). \tag{1.92}$$

We will now put ourselves in the simplest case, which is putting ourselves in the equatorial plane $\theta = \pi/2$. The Lagrangian will then be

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = \frac{1}{2}[A(r)c^2\dot{t}^2 - B(r)\dot{r}^2 - r^2\dot{\phi}^2], \tag{1.93}$$

where the factors with θ are zero when they appear in a time derivative and one when they appear in the sine. We can now see that the t and ϕ coordinate are cyclic, i.e. they do not appear in the Lagrangian, and this implies that their conjugate momenta

$$\frac{d\mathcal{L}}{d\dot{t}} = A(r)c^2\dot{t} = D(r)c^2 = const., \tag{1.94}$$

$$\frac{d\mathcal{L}}{d\dot{\phi}} = -r^2\dot{\phi} = -h = const., \tag{1.95}$$

are conserved quantities.
Since pulsar signals are basically photons, we will have to study the free-falling motion of a massless particle. This implies that

$$\mathcal{L} = A(r)c^2\dot{t}^2 - B(r)\dot{r}^2 - r^2\dot{\phi}^2 = 0. \tag{1.96}$$

Now, using 1.95, we get

$$A(r)c^2\dot{t}^2 - B(r)\dot{r}^2 - \frac{h^2}{r^2} = 0. \tag{1.97}$$

If we compute this in the case of the distance of closest approach r_0, we will have that $\dot{r} = 0$, so

$$A(r_0)c^2\dot{t}^2 = \frac{h^2}{r_0^2} \implies \frac{h^2}{C^2(r_0)} = \frac{r_0^2c^2}{A(r_0)}, \tag{1.98}$$

where we have written $C^2(r_0) = A^2(r_0)\dot{t}^2$. We can then compute what we are looking for, which is $\left(\dfrac{dr}{dt}\right)^2$. In order to do that let's start by dividing 1.97 by $\dot{t}$

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{\dot{r}}{\dot{t}}\right)^2 = \frac{A(r)c^2}{B(r)} - \frac{h^2}{\dot{t}^2 r^2 B(r)}, \tag{1.99}$$

and considering the following identity, which comes from the previous definitions

$$\frac{r^2\dot{\phi}^2}{\dot{t}^2} = \frac{h^2}{r^2}A^2(r)\frac{\dot{t}}{D^2(r)c^2}. \tag{1.100}$$

By using 1.98, it becomes

$$\frac{r^2\dot{\phi}^2}{\dot{t}^2} = \frac{A^2(r)}{r^2}\frac{r_0^2}{A(r_0)}. \tag{1.101}$$

Putting this in 1.100 we finally have

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{\dot{r}}{\dot{t}}\right)^2 = \frac{A(r)c^2}{B(r)} - \frac{r^4\dot{\phi}^2}{\dot{t}^2 r^2 B(r)} = \frac{A(r)c^2}{B(r)} - \frac{A^2(r)}{A(r_0)}\frac{r_0^2}{B(r)r^2}, \tag{1.102}$$

which can be rewritten in the more compact form

$$\left(\frac{\dot{r}}{\dot{t}}\right)^2 = c^2\frac{A(r)}{B(r)}\left[1 - \frac{A(r)}{A(r_0)}\frac{r_0^2}{r^2c^2}\right]. \tag{1.103}$$

From here it's very easy to get to $\left(\dfrac{dt}{dr}\right)^2$, and then all that is left to do is to integrate (we will put $c = 1$ for simplicity), so

$$F(r_1) = \int_{r_0}^{r_1}\left(\frac{B(r)}{A(r)}\right)^{1/2}\left[1 - \frac{A(r)}{A(r_0)}\frac{r_0^2}{r^2}\right]^{-1/2} dr. \tag{1.104}$$

Now that we have a general formula that can be used for every spherical symmetric metric, let us use the most appropriate one for the GR case, which is the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right)c^2dt^2 - \frac{dr^2}{\left(1 - \dfrac{2m}{r}\right)} - r^2(d\phi^2 + \sin^2\theta d\theta), \tag{1.105}$$

where

$$2m = r_s = \frac{2GM}{c^2}, \tag{1.106}$$

is the Schwarzschild radius. Let's now put the coefficent in the integral formula

$$F(r_1) = \int_{r_0}^{r_1} \frac{dr}{\left(1 - \dfrac{2m}{r}\right)} \left[1 - \frac{\left(1 - \dfrac{2m}{r}\right)}{\left(1 - \dfrac{2m}{r_0}\right)} \frac{r_0}{r^2}\right]^{-1/2}. \tag{1.107}$$

In order to solve this integral we will expand the integrand to the linear order in the Schwarzschild radius

$$\frac{1}{\left(1 - \dfrac{2m}{r}\right)} \left[1 - \frac{\left(1 - \dfrac{2m}{r}\right)}{\left(1 - \dfrac{2m}{r_0}\right)} \frac{r_0}{r^2}\right]^{-1/2} \simeq \frac{r}{\sqrt{r^2 - r_0^2}} \left[1 + \frac{m(2r + 3r_0)}{r(r + r_0)}\right] + \mathcal{O}(m^2). \tag{1.108}$$

If we, then, evaluate the integral, we obtain

$$F(r_1) = \sqrt{r_1^2 - r_0^2} + 2m \ln\left(\frac{r_1 + \sqrt{r_1^2 - r_0^2}}{r_0}\right) + m\sqrt{\frac{r_1^2 - r_0^2}{r_1^2 + r_0^2}} + \mathcal{O}(m^2). \tag{1.109}$$

Now, in order to get the Shapiro time delay we have to subtract the non relativistic time delay t_{NR} from the relativistic one. This way we will get, after some computation

$$\begin{aligned} &c\Delta t_{Shapiro} = c(t_R - t_{NR}) = \\ &2m \ln\left[\frac{(r_1 + \sqrt{r_1^2 - r_0^2})(r_2 + \sqrt{r_2^2 - r_0^2})}{r_0^2}\right] + m\left(\sqrt{\frac{r_1^2 - r_0^2}{r_1^2 + r_0^2}} + \sqrt{\frac{r_2^2 - r_0^2}{r_2^2 + r_0^2}}\right). \end{aligned} \tag{1.110}$$

This is the formula for the Shapiro time delay in the case of $\theta = \pi/2$. Let's now look at the general case where $\theta \neq \pi/2$. Let us use another approach, which is similar to the classical one[7], and let us start from

$$\Delta_S = -\frac{2}{c} \int_{\boldsymbol{x_p}(t_e)}^{\boldsymbol{x_b}(t_a)} |d\boldsymbol{x}| \phi(\boldsymbol{x}), \tag{1.111}$$

where t_e will be the time of emission, t_a will be the time of arrival, and $\boldsymbol{x_b}, \boldsymbol{x_p}, \boldsymbol{x_c}$ will be the positions of the Earth-Sun barycenter, the pulsar and the companion respectively. This expression can be expanded and manipulated in order to obtain

$$\Delta_S \approx \frac{2Gm_c}{c^3} \int_{t_e}^{t_a} \frac{dt}{|\boldsymbol{y}(t) - \boldsymbol{x_2}(t_e)|} + const, \tag{1.112}$$

where $\boldsymbol{y(t)}$ is a straight coordinate path from $\boldsymbol{x_1}(te)$ to $\boldsymbol{x_b}(t_a)$

$$\boldsymbol{y}(t) = \boldsymbol{x_1}(t_e) + \frac{t - t_e}{t_a - t_e}(\boldsymbol{x_b}(t_a) - \boldsymbol{x_1}(t_e)). \tag{1.113}$$

Let $\boldsymbol{x} \equiv \boldsymbol{x_1}(t_e) - \boldsymbol{x_2}(t_e)$ and $\theta = \dfrac{t - t_e}{t_a - t_e}$ then, except for an unobservable constant, we will obtain

$$\Delta_S(t_e) = \frac{2Gm_c}{c^3}(t_a - t_e)\int_0^1 \frac{d\theta}{|\boldsymbol{x} + \theta(\boldsymbol{x_b}(t_a) - \boldsymbol{x_1}(t_e))|}. \tag{1.114}$$

Now, since$|\boldsymbol{x_b}(t_a) - \boldsymbol{x_1}(t_e)| \simeq t_a - t_e$ and $|\boldsymbol{x_b}(t_a)| \simeq |\boldsymbol{x_b}(0)| =: r_b \gg |\boldsymbol{x_1}(t_e)|$ we will have

$$\Delta_s(t_e) \simeq \frac{2Gm_c}{c^3}\int_0^1 \frac{d\theta}{|\theta\boldsymbol{n} + \frac{\boldsymbol{x}}{r_b}|} = \frac{2Gm_c}{c^3}\int_0^1 \frac{d\theta}{\sqrt{\theta^2 + \theta\boldsymbol{n}\cdot\frac{\boldsymbol{x}}{r_b} + (\frac{r}{r_b})^2}}, \tag{1.115}$$

where $r := |\boldsymbol{x}|$ and $\boldsymbol{n}$ is the unit vector along the line of sight. We can then integrate and we will obtain

$$\begin{aligned}\Delta_S(t_e) &\simeq \frac{2Gm_c}{c^3}\ln\left(\frac{2r_b}{r_b + \boldsymbol{n}\cdot\boldsymbol{x}}\right) + const = \\ &-\frac{2Gm_c}{c^3}\ln(\boldsymbol{n}\cdot(\boldsymbol{x_1}(t_e) - \boldsymbol{x_2}(t_e)) + |\boldsymbol{x_1}(t_e) - \boldsymbol{x_2}(t_e)|) + const.\end{aligned} \tag{1.116}$$

We can use the keplerian approximation on the right hand side and this implies that the argument of the logarithm will be

$$r(1 - \sin i \sin(\phi + \omega)) = r(1 - \sin i(\sin\omega\cos\phi + \cos\omega\sin\phi)), \tag{1.117}$$

where ω is the periastron angle and phi is a funtion of the eccentric anomaly.
We can now manipulate he formula further by using the following expressions

$$\cos\phi = \frac{\cos u - e}{1 - e\cos u}, \tag{1.118}$$

$$\sin\phi = \frac{\sqrt{1 - e^2}\sin u}{1 - e\cos u}, \tag{1.119}$$

$$r = a(1 - e\cos u), \tag{1.120}$$

to obtain the final expression

$$\Delta_S(t_e) = -\frac{2Gm_c}{c^3}\ln\Big(1 - e\cos u - \sin i(\sin\omega(\cos u - e) + \sqrt{1 - e^2}\cos\omega\sin u)\Big). \tag{1.121}$$

This formula is also often written as

$$\Delta_S(t_e) = -2r\ln\Big(1 - e\cos u - s(\sin\omega(\cos u - e) + \sqrt{1-e^2}\cos\omega\sin u)\Big), \tag{1.122}$$

where r and s are defined respectively as

$$r = \frac{2Gm_c}{c^3}, \tag{1.123}$$

$$s = \sin i, \tag{1.124}$$

and are called ”range“ and ”shape“ of the Shapiro delay.

Chapter 2

Einstein's equations and $f(R)$ theories

One hundred years after the introduction of the theory of General Relativity (GR), questions related to its limitations are becoming come and more pertinent [8]. From a theoretical point of view, the first modifications to GR came to be beacuse of the fact that this is a non renormalizable theory, which means that it cannot be conventionally quantized. It was later shown that renormalization at one-loop demands that the Einstein-Hilbert action be supplemented by higher order curvature terms. A more recent and observation-related motivation for an extension of GR is the problem of dark matter, an unkown form of matter, which has the clustering properties of ordinary matter but has not yet been detected in the laboratory, and dark energy an unknown form of energy which not only has not been detected directly, but also does not cluster as ordinary matter does. The simplest model wich fits the data is the Lambda Cold Dark Matter model (ΛCDM) supplemented by an inflactionary scenario, but this is more of an empyrical fit and it theoretical motivation can be regarded as quite poor. All of these problems spark the question: could it be that our description of the gravitational interaction at the relevant scales is not sufficiently adequate and stands at the root of all or some of these problems? If that was so, a way to try and solve the problems could be to modify our theory of gravity.

2.1 Einstein's equations

Before presenting modified theories of gravity, it is useful to see hoew to get to Einstein's field equations [9]. The derivation we will show here it's not Einstein's one, since that one only comes from physical considerations. We will rather use the Einstein-Hilbert action, since, even if this is a purely mathematical way, it is the most rigorous way to derive the equations. So let us start by writing the action

$$S_{EH} = \int \sqrt{-g} d^4x \left(\frac{c^4}{16\pi G} R + L_M \right), \tag{2.1}$$

where g is the determinant of the metric, R is the Ricci scalar and L_M is the matter Lagrangian.

Now let us make a little variation respect to the metric tensor $g^{\mu\nu}$

$$\delta S_{EH} = \int d^4x \left(\frac{c^4}{16\pi G} \frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}L_M)}{\delta g^{\mu\nu}} \right) \delta g^{\mu\nu}, \tag{2.2}$$

and use Leibniz's rule to expand it

$$\delta S_{EH} = \int \sqrt{-g} d^4x \left(\frac{c^4}{16\pi G} \left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_M)}{\delta g^{\mu\nu}} \right) \delta g^{\mu\nu}. \tag{2.3}$$

We will now focus on the first term, and we will us the definition of the Ricci scalar as the contraction between the metric tensor and the Ricci tensor

$$\frac{\delta R}{\delta g^{\mu\nu}} = \frac{\delta(g^{\mu\nu}R_{\mu\nu})}{\delta g^{\mu\nu}} = R_{\mu\nu} \frac{\delta g^{\mu\nu}}{\delta g^{\mu\nu}} + g^{\mu\nu} \frac{R_{\mu\nu}}{\delta g^{\mu\nu}} = R_{\mu\nu} + g^{\mu\nu} \frac{R_{\mu\nu}}{\delta g^{\mu\nu}}. \tag{2.4}$$

It can be shown that the last piece in the above equation can be expressed as a total derivative, which means that it gives no contribution to the variation of the functional and can be discarded.

We will then look at the second term. To compute this we have to transform the coordinate system to one where the metric tensor is diagonal and then we have to apply Leibniz's rule

$$\frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \frac{-1}{\sqrt{-g}} (-1) g g^{\mu\nu} \frac{\delta g^{\mu\nu}}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu} R. \tag{2.5}$$

Lastly, we define the last term as the stress-energy tensor

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_M)}{\delta g^{\mu\nu}} = -\frac{1}{2} T_{\mu\nu} \tag{2.6}$$

We will have that the least action is obtained when the integrand in 2.3 is zero, so

$$\frac{c^4}{16\pi G} \left(R_{\mu,\nu} - -\frac{1}{2} g_{\mu\nu} R \right) - \frac{1}{2} T_{\mu\nu} = 0, \tag{2.7}$$

or, casting them in the most well known form,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.8}$$

These are Einstein's field equations.

2.2 $f(R)$ theories

All of $f(R)$ theories come from a generalization of the Lagrangian in the Einstein-Hilbert action, which means that rather than having

$$S_{EH} = \frac{1}{2\mathcal{K}} \int d^4x \sqrt{-g}\, R, \tag{2.9}$$

where $\mathcal{K} = 8\pi G$, we will have

$$S_{EH} = \frac{1}{2\mathcal{K}} \int d^4x \sqrt{-g}\, f(R). \tag{2.10}$$

This particular choice could spark some questions. First of all, we could ask ourselves why we use $f(R)$ actions and not more general ones, like something including higher order invariants like $R_{\mu\nu}R^{\mu\nu}$. There is more than one reason for this. The first one is that of simplicity: $f(R)$ theories are sufficiently general to enclose some of the basic characteristics of higher order gravity, but at the same time simple enough to be handled easily. As an example, if we view f as a series expansion

$$f(R) = \ldots + \frac{\alpha_2}{R^2} + \frac{\alpha_1}{R} - 2\Lambda + R + \frac{R^2}{\beta_2} + \frac{R^3}{\beta_3} + \ldots, \tag{2.11}$$

where α_i and β_i have the appropriate dimensions, we can see that it contains a number of phenomenologically interesting terms. The second reason is that $f(R)$ theories seem to be special in the fact that they are the only one to avoid the Ostrogradski instability[1]. We must always keep in mind, though, that $f(R)$ gravity must neither be over or underestimated. While there are already studies that brought to useful conclusion, we should remember that $f(R)$ theories are just an easy-to-handle deviation from Einstein's theory, mostly to be used in order to understand the principles and limitations of modified gravity. Getting to the action and the field equations, we have to point out that there are actually two variational principles that we can use to get to Einstein's equation: the standard metric variation and the Palatini variation. The main feature of the latter is that the metric and the connection are both assumed to be independent variables and one varies the action with respect to both of them under the assumption tha the the matter action does not depend on the connection. While in tha case of $f(R) = R$

[1]The Ostrogradski instabilityis a consequence of a theorem of Michael Ostrogradski in classical mechanics wich states that a non-degenerate Lagrangian dependent on time derivatives higher than the first corresponds to a linearly unstable Hamiltonian associated with the Lagrangian via a Legendre transformation. This instability has also been proposed as an expanation as to way no differential equations of order higher than two appear to describe a physicial phenomenon. [10]

the two variational principles lead to the same equations, this is not true for the general case. The choice of variational principle is referred to as "formalism" and this implies that there will be two different versions of $f(R)$ gravity: metric $f(R)$ gravity and Palatini $f(R)$ gravity. Actually, there is also a third version of $f(R)$ gravity which is called metric-affine $f(R)$ gravity in which one uses the Palatini variation but abandons the assumption that the matter action is independent of the connection. This is, indeed, the most general case.

2.2.1 Metric Formalism

Let us start form the action 2.10 and add the matter action. We will the get the total $f(R)$ action

$$S_{met} = \frac{1}{2\mathcal{K}} \int d^4x\sqrt{-g}\, f(R) + S_M(g_{\mu\nu}, \psi), \tag{2.12}$$

where ψ represents all of the matter fields. Doing the variation with respect to the metric and manipulating the result we get

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]f'(R) = \mathcal{K}T_{\mu\nu}, \tag{2.13}$$

where

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{2.14}$$

These are fourth order partial differential equations in the metric, since R already contains the second derivative of the latter. It can be easily seen that for an action that is linear in R the fourth order terms vanish and we get standard GR. It is interesting to look at the trace of 2.13

$$f'(R)R - 2f(R) + 3\Box f' = \mathcal{K}T, \tag{2.15}$$

where $T = g_{\mu\nu}T^{\mu\nu}$. We can see that, while for GR the relation between R and T is algebraic ($R = -\mathcal{K}T$), for $f(R)$ gravity R and T are related differentially. This is a good indication that $f(R)$ theories admit a larger variety of solutions than standard GR. As an example, we have that the Jebsen-Birkhoff's theorem, stating that the Schwarzschild solution is the unique spherically symmetric vacuum solution, is no longer valid in metric $f(R)$ gravity. Without going into details, let us stress the fact that $T = 0$ no longer implies that $R = 0$, or is even constant. We can also use 2.15 to make some remarks on maximally symmetric solutions. Recalling that this kind of solutions lead to a constant Ricci scalar, we check the case in which $R = const.$ and $T = 0$, for which 2.13 becomes

$$f'(R)R - 2f(R) = 0 \tag{2.16}$$

which, for a given f, is just an algebraic equation in R. If $R = 0$ is a root of this equation, and one takes this root, then equation 2.13 reduces to $R_{\mu\nu} = 0$ and hence the maximally symmetric solution is Minkowsky spacetime. On the other hand, if the root of is $R = C$, where C is a generic constant, then 2.13 reduces to $R_{\mu\nu} = g_{\mu\nu}C/4$and the maximally symmetric solution is de Sitter or anti-de Sitter space depending on the sign of C, just as in GR with a cosmological constant. Finally, it is interesting to note that the field equations can be written in the form of Einstein's equations with an effective stress-energy tensor with curvature terms moved to the right hand side. While this is questionable, since this is not Einstein's theory and forcing upon it an interpretation in terms of Einstein's equations is artificial, it is shown that in can be useful in scalar-tensor gravity. The form of the field equation will be

$$G_{\mu\nu} = \frac{\mathcal{K}T_{\mu\nu}}{f'(R)} + g_{\mu\nu}\frac{[f(R) - Rf'(R)]}{2f'(R)} + \frac{\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)}{f'(R)}$$
$$= \frac{\mathcal{K}}{f'(R)}(T_{\mu\nu} + T^{(eff)}_{\mu\nu}), \quad (2.17)$$

where we can also define $G_{eff} = G/f'(R)$ as the effective gravitational coupling strength in analogy to what is done in scalar-tensor gravity. This is also useful because the effective stress-energy tensor can be put in the form of a perfect fluid energy-momentum tensor [8].

2.2.2 Palatini Formalism

Let's now present the Palatini Formalism, which is an independent variation of the action with respect to the metric and an independent connection $\Gamma^\lambda{}_{\mu\nu}$. The action will formally be the same, but the Ricci tensor and Ricci scalar will be the one built from the independent connection. To make things clear, let us define the Ricci tensor in the Palatini formalism as $\mathcal{R}_{\mu\nu}$, and the corresponding Ricci scalar as $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$. The action will then have the form

$$S_{met} = \frac{1}{2\mathcal{K}}\int d^4x\sqrt{-g}\, f(\mathcal{R}) + S_M(g_{\mu\nu}, \psi), \quad (2.18)$$

where we have made the fundamental assumption that the matter action S_M depends only on the metric and the matter fields and not on the independent connection. This assumption has consequences in the physical meaning of the connection. Recall that an affine connection usually defines parallel transport and the covariant derivative; we also know that the matter action should be a generally covariant scalar which includes the derivatives of the matter fields. This means that the these derivatives should be covariant derivatives for a general matter field.

Exceptions, like scalar fields, exist, but we must remember that S_M should include all of the possible fields. Having said that, we have that, assuming that S_M should contain all possible fields, two options arise:

1. We are restricting ourselves to specific fields;

2. We are implictly assuming that parallel transport is defined with the Levi-Civita connection of the metric.

Since the first option is very limiting for a gravitational theory, we get to the conclusion that parallel transport and the covariant derivative are not defined by the independent connection $\Gamma^{\lambda}_{\ \mu\nu}$, but by the Levi-Civita one $\{^{\ \lambda}_{\mu\nu}\}$.
We can now look at the field equations. If we vary the action independently with respect to the metric and the independent connection, and using the following formula

$$\delta\mathcal{R}_{\mu\nu} = \bar{\nabla}_{\lambda}\Gamma^{\lambda}_{\ \mu\nu} - \bar{\nabla}_{\nu}\Gamma^{\lambda}_{\ \mu\lambda}, \tag{2.19}$$

we get

$$f'(\mathcal{R})\mathcal{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \mathcal{K}T_{\mu\nu}, \tag{2.20}$$

$$-\bar{\nabla}_{\lambda}(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) + \bar{\nabla}_{\sigma}(\sqrt{-g}f'(\mathcal{R})g^{\sigma(\mu})\delta^{\nu)}_{\mu} = 0, \tag{2.21}$$

where $\bar{\nabla}$ stands for the covariant derivative defined with the independent connection and $(\mu\nu)$, $[\mu\nu]$ denote symmetrization and antisymmetrization over the indeces μ, ν, respectively. If we then take the trace of 2.21, we get

$$\bar{\nabla}_{\sigma}(\sqrt{-g}f'(\mathcal{R})g^{\sigma\mu}) = 0. \tag{2.22}$$

This can be used to express the field equations as

$$f'(\mathcal{R})\mathcal{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \mathcal{K}T_{\mu\nu}, \tag{2.23}$$

$$\bar{\nabla}_{\lambda}(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) = 0. \tag{2.24}$$

We can now easily see how the Palatini formalism gives back GR in the case of $f(\mathcal{R}) = \mathcal{R}$: $f'(\mathcal{R})$ will be equal to one and 2.24 becomes the definition of the Levi-Civita connection; these conditions imply that $\mathcal{R}_{\mu\nu} = R_{\mu\nu}$ and $\mathcal{R} = R$, which, in turn, means that 2.23 gives back Einstein's equations. We can note that, in the Palatini Formalism, the fact that the connection turns out to be the Levi-Civita one comes directly from the equations rather than from an a priori assumption.
Let's now look at some useful manipulations of the field equations. We start by takin the trace of 2.23

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \mathcal{K}T. \tag{2.25}$$

For a given f, 2.25 it's an algebraic equation in $\mathcal{R}$. For the cases in which $T = 0$, $\mathcal{R}$ will be a constant and a root of the equation

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 0. \tag{2.26}$$

The cases for which this equation has no roots should not be considered, since it can be shown that the field equations are inconsistent [11]. It can be shown that the equation can be satisfied by $f(\mathcal{R}) \propto \mathcal{R}^2$ and that this particular choice for f leads to a conformally invariant theory [11]. This solution, while valid, cannot be considered for a low energy theory of gravity, since it requires special conditions on gravity that are not generally met.

Let us consider now 2.24. We start by defining a metric which is conformal[2] to $g_{\mu\nu}$ as

$$h_{\mu\nu} \equiv f'(\mathcal{R})g_{\mu\nu}. \tag{2.27}$$

It can be easily shown that [3]

$$\sqrt{-h}h^{\mu\nu} = \sqrt{-g}f'(\mathcal{R})g^{\mu\nu}. \tag{2.28}$$

This implies that 2.24 becomes the definition of the Levi-Civita connection for $h_{\mu\nu}$ and can be solved to give

$$\Gamma^{\lambda}_{\ \mu\nu} = \frac{1}{2}h^{\lambda\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}). \tag{2.29}$$

Having that 2.25 algebraically relates $\mathcal{R}$ to T and that we can express $\Gamma^{\lambda}_{\ \mu\nu}$ as an explicit combination of $\mathcal{R}$ and $g^{\mu\nu}$, we can, in principle, eliminate the independent connection from the equations and express the latter only in terms of the metric and the matter fields. The fact that we can express $\Gamma^{\lambda}_{\ \mu\nu}$ as an explicit function of $\mathcal{R}$ and $g^{\mu\nu}$ indicates that the former can be seen as some sort of auxiliary field [8]. For the moment, let us take into account how the Ricci tensor transforms under-conformal transformations and write

$$\mathcal{R}_{\mu\nu} = R + \frac{3}{2}\frac{1}{(f'(\mathcal{R}))^2}(\nabla_\mu f'(\mathcal{R}))(\nabla_\nu f'(\mathcal{R})) - \\ - \frac{1}{f'(\mathcal{R})}\left(\nabla_\mu\nabla_\nu - \frac{1}{2}g_{\mu\nu}\Box\right)f'(\mathcal{R}) \tag{2.30}$$

[2]Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be open subets of $\mathbb{R}^n$. A function $\boldsymbol{f} : \boldsymbol{U} \to \boldsymbol{V}$ is called conformal (or angle-preserving) at a point $u_0 \in \boldsymbol{U}$if it preserves angles between directed curves through u_0 as well as preserving orientation. Conformal maps preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size or curvature.

[3]We have to point out that this computation is only valid in 4 dimensions. For the general case the definition of the metric conformal to $g_{\mu\nu}$ should be $h_{\mu\nu} \equiv [f'(\mathcal{R})]^{2/D-2}g_{\mu\nu}$, where D is the number of dimensions.

By then contracting it with $g^{\mu\nu}$ we get

$$\mathcal{R} = R + \frac{3}{2}\frac{1}{(f'(\mathcal{R}))^2}(\nabla_\mu f'(\mathcal{R}))(\nabla^\mu f'(\mathcal{R})) + \frac{3}{f'(\mathcal{R})}\Box f'(\mathcal{R}). \tag{2.31}$$

It is important to note the difference between $\mathcal{R}$ and the Ricci scalar of $h_{\mu\nu}$, which is due to the the fact that we have used $g_{\mu\nu}$ for the contraction with $\mathcal{R}_{\mu\nu}$. Let us now substitute 2.30 and 2.31 into equation 2.23; after some manipulations we will get

$$G_{\mu\nu} = \frac{\mathcal{K}}{f'}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(\mathcal{R} - \frac{f}{f'}) + \frac{1}{f'}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f' - \\ - \frac{3}{2f'^2}\left[(\nabla_\mu f')(\nabla_\nu f') - \frac{1}{2}g_{\mu\nu}(\nabla f')^2\right]. \tag{2.32}$$

If we assume that we know the root of 2.25, we have completely eliminated the independent connection from the equation. This means that we have effectively reduced the number of equations to one and it can also be noticed that all quantities on both sides of 2.32 depend only on the metric and the matter fields. We have brought the theory in a form that looks like GR, but with a modified source.
What we can deduce from all of this is the following:

1. When $f(\mathcal{R}) = \mathcal{R}$ the theory reduces to GR;
2. For matter fields with $T = 0$, the theory reduces to GR with the presence of a cosmological constant and a modified coupling constant. The expression of the cosmological constant can then be obtained from 2.26. It's useful to remember that, beside vacuum, $T = 0$ also for electromgnetic fields, radiation and any other type of conformally invariant matter;
3. In the general case of $T \neq 0$ the modified source includes derivatives of the stress-energy tensor[4], unlike in GR. This is because of the fact that, since f' is a function of $\mathcal{R}$ and $\mathcal{R}$ is a function of T (see 2.25).

2.2.3 Metric-affine formalism

While discussing Palatini formalism we have stated that the independent connection assumed the role of an auxiliary field while the connection that carries the usual geometical meaning was still the Levi-Civita one. If we, instead, keep true

[4]Except in special cases such as a perfect fluid, $T_{\mu\nu}$ and consequently T already include first derivatives of the matter fields. This means that equation 2.32 will include at least second derivatives of the matter fields.

to the geometrical meaning of the independent connection $\Gamma^{\lambda}{}_{\mu\nu}$, we would have to define the covariant derivative in the matter action by using the independent connection. This is just what is done in the metric affine formalism, in which $S_M = S_M(g_{\mu\nu}, \psi, \Gamma^{\lambda}{}_{\mu\nu})$. The resulting action will then be

$$S_{ma} = \frac{1}{2\mathcal{K}} \int d^4x \sqrt{-g}\, f(\mathcal{R}) + S_M(g_{\mu\nu}, \psi, \Gamma^{\lambda}{}_{\mu\nu}). \tag{2.33}$$

Now, before presenting the metric-affine formalism in detail, there are certain issues that must be addressed. First of all, since the matter action now depends on the connection, we should define a quantity that represent the variation of S_M with respect to the connection, which mimics the definition of the stress-energy tensor. This quantity is called hypermomentum and is defined as [12]

$$\Delta_{\lambda}^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta \Gamma^{\lambda}{}_{\mu\nu}}. \tag{2.34}$$

Since the connection has now the role of an auxiliary field, it can be interesting to see what happens when we drop all the assumptions that we usually make about it. In particular, let us drop the assumptions that the connection is related to the metric and that it is symmetric. Since we have done that, it is useful to define:

1. a non-metricity tensor
$$Q_{\mu\nu\lambda} \equiv -\bar{\nabla}_{\mu} g_{\nu\lambda}, \tag{2.35}$$
which measures the failure of the connection to covariantly conserve the metric;

2. The trace of the non-metricity tensor with respect to the last two indices, which is also called Weyl vector
$$Q_{\mu} \equiv \frac{1}{4} Q^{\nu}_{\mu\nu}; \tag{2.36}$$

3. The Cartan torsion tensor
$$S^{\lambda}_{\mu\nu} \equiv \Gamma^{\lambda}_{[\mu\nu]}, \tag{2.37}$$
which is the antisymmetric part of the connection.

In the case of a non-vanishing Cartan tensor, the theory will necessarily include torsion [5] Unfortunately, our choice of leaving the connection without any constric-

[5] In general, on a differentiable manifold equipped with an affine connection, torsion is one of the two fundamental invariants of the connection and gives an intrinsic characterization of how tangent spaces twist about a curve when they are parallel transported; it is complemented by the other invariant of the connection, the curvature, which describes how the tangent spaces roll along the curve.

tion comes with a complication. Let us consider the projective transformation[6]

$$\Gamma^{\lambda}_{\ \mu\nu} \to \Gamma^{\lambda}_{\ \mu\nu} + \delta^{\lambda}_{\ \mu}\xi_{\nu}, \tag{2.38}$$

where ξ_{ν} is a generic covariant vector field. It can be shown that the Ricci tensor will transform like

$$\mathcal{R}_{\mu\nu} \to \mathcal{R}_{\mu\nu} - 2\partial_{[\mu}\xi_{\nu]}. \tag{2.39}$$

However, because of the fact that the metric is symmetric, the curvature scalar does not change

$$\mathcal{R} \to \mathcal{R}, \tag{2.40}$$

which, of course, implies that $\mathcal{R}$ is invariant under projective transformations. This further implies that any action built from functions of $\mathcal{R}$ will be invariant under projective transformations. The problem arise when we consider the matter action, since it isn't generically projective ivariant and this would lead to inconsistencies in the field equations. The solution could be found by generalizing the action and making so that it breaks projective invariance, but if we want to remain in the framework of $f(R)$ gravity the choice to make is only one: we have to constrain the connection. In order to do this is useful to analyze what projective invariance means. The concept is very similar to that of gauge invariance in electromagnetism, in the sense that it tells us that the field (in this case $\Gamma^{\lambda}_{\ \mu\nu}$) can be obtained from the field equations up to a projective transformation. Continuing the similarity with gauge invariance, we can simply break projective invariance by fixing some degrees of freedom, whose number will be equal to the number of components involved in the transformation, *i.e.*, four. We also know from 2.26 that a symmetric connection would break projective invariance, which means that the fixing has to be done on the non-symmetric part of the connection. This can be achieved [13] by demanding that $S_{\mu} = S^{\sigma}_{\sigma\mu}$ be equal to zero, which works for both a linear and an $f(R)$ action. This costrain can be imposed by adding a Lagrange multiplier B^{μ}. There will, then, be an additional term in the action in the form

$$S_{LM} = \int d^4x\sqrt{-g}B^{\mu}S_{\mu}. \tag{2.41}$$

We finally have the most general action in the metric-affine formalism, which is

$$S_{ma} = \frac{1}{2\mathcal{K}}\int d^4x\sqrt{-g}\, f(\mathcal{R}) + S_M(g_{\mu\nu}, \psi, \Gamma^{\lambda}_{\ \mu\nu}) + S_{LM}. \tag{2.42}$$

Let's now get to the field equations. In order to do this, we have to vary independently with respect to the metric, the connection and the Lagrange multiplier;

[6]A projective transformation, or homography, is an isomorphism of projective spaces. This isomorphism is a bijection that maps lines to lines.

the procedure to get the equations is computation-heavy, so we will just write the important formulas needed for the computation. First of all, we need the variation

$$\delta \mathcal{R}_{\mu\nu} = \bar{\nabla}_\lambda \delta \Gamma^\lambda{}_{\mu\nu} - \bar{\nabla}_\nu \delta \Gamma^\lambda{}_{\mu\lambda} + 2\Gamma^\sigma{}_{[\nu\lambda]} \delta \Gamma^\lambda{}_{\mu\sigma}, \tag{2.43}$$

where we have to stress that the covariant derivative is defined with respect to the independent connection

$$\bar{\nabla}_\mu a^\nu{}_\sigma = \partial_\mu a^\nu{}_\sigma + \Gamma^\nu{}_{\mu\alpha} A^\alpha{}_\sigma - \Gamma^\alpha{}_{\mu\sigma} A^\nu{}_\alpha, \tag{2.44}$$

and the same is valid for the Ricci tensor

$$\mathcal{R}_{\mu\nu} = \partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\mu\lambda} + \Gamma^\lambda{}_{\sigma\lambda} \Gamma^\sigma{}_{\mu\nu} - \Gamma^\lambda{}_{\sigma\nu} \Gamma^\sigma{}_{\mu\lambda}. \tag{2.45}$$

The end result will be

$$f'(\mathcal{R}) \mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \mathcal{K} T_{\mu\nu}, \tag{2.46}$$

$$\mathcal{K}(\Delta_\lambda^{\mu\nu} - B^{[\mu} \delta^{\nu]}{}_\lambda) = \frac{1}{\sqrt{-g}} [-\bar{\nabla}_\lambda (\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}) + \bar{\nabla}_\sigma (\sqrt{-g} f'(\mathcal{R}) g^{\mu\sigma}) \delta^\nu{}_\lambda] + \\ + 2 f'(\mathcal{R}) (g^{\mu\nu} S^\sigma{}_{\lambda\sigma} - g^{\mu\rho} S^\sigma{}_{\rho\sigma} \delta^\nu{}_\lambda + g^{\mu\sigma} S^\nu{}_{\sigma\lambda}), \tag{2.47}$$

$$S^\sigma{}_{\mu\sigma} = 0. \tag{2.48}$$

By taking the trace of 2.47 over the indices μ and λ, and using 2.48, we get

$$B^\mu = \frac{2}{3} \Delta_\sigma^{\sigma\mu}. \tag{2.49}$$

This bring us to the final form of the equations, which is

$$f'(\mathcal{R}) \mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \mathcal{K} T_{\mu\nu}, \tag{2.50}$$

$$\mathcal{K}(\Delta_\lambda^{\mu\nu} - \frac{2}{3} \Delta_\sigma^{\sigma[\mu} \delta^{\nu]}{}_\lambda) = \frac{1}{\sqrt{-g}} [-\bar{\nabla}_\lambda (\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}) + \bar{\nabla}_\sigma (\sqrt{-g} f'(\mathcal{R}) g^{\mu\sigma}) \delta^\nu{}_\lambda] + \\ + 2 f'(\mathcal{R}) (g^{\mu\nu} S^\sigma{}_{\lambda\sigma} - g^{\mu\rho} S^\sigma{}_{\rho\sigma} \delta^\nu{}_\lambda + g^{\mu\sigma} S^\nu{}_{\sigma\lambda}), \tag{2.51}$$

$$S^\sigma{}_{\mu\sigma} = 0. \tag{2.52}$$

Let us now examine the role of $\Delta_\lambda^{\mu\nu}$. By splitting 2.51 in symmetric and antisymmetric part, and using some manipulations and contractions, it can be shown that [14]

$$\Delta_\lambda^{[\mu\nu]} = 0 \implies S_{\mu\nu}{}^\lambda = 0. \tag{2.53}$$

This implies that any torsion is introduced by matter fields that have $\Delta_\lambda^{[\mu\nu]} \neq 0$ and also that torsion is not propagating, since it is given algebraically in terms of $\Delta_\lambda^{[\mu\nu]}$. This means that tortion can only be detected in the presence of such matter fields and, in absence of those, there will be no tortion.
In the same way, we can use the symmetrized version of 2.51 to show that the symmetric part of the hypermomentum $\Delta_\lambda^{(\mu\nu)}$ is algebraically related to the non metricity $Q_{\lambda\mu\nu}$. Therefore, matter fields with non-zero $\Delta_\lambda^{(\mu\nu)}$ will introduce non-metricity, even though we have to underline the fact that things are more complicated, since non-metricity is also partly due to the functional form of the Lagrangian [14].
We will now briefly examine the properties of $\Delta_\lambda^{\mu\nu}$ in terms of specific fields[14]. For this purpose, we will then need the action of the matter field in curved space-time. We know that, in a purely metric theory, any covariant equation, and hence also the action, can be written in a local inertial frame by assuming that the metric is flat and that the connection vanishes. One expects, of course, that the inverse procedure, which is the minimal coupling principle, should hold as well and can be used to get the expression of the action in curved spacetime starting by its expression in a local inertial frame. This expectation comes from the conjecture that the components of the gravitational field should be used in the matter on a necessity basis. This conjecture can be stated in GR in the following form: the metric should be used in the matter action only for contracting indices and constructing the terms that need to be added in order to write a viable covariant matter action. This implies that the connection should never appear alone in the action, which makes sense since the connection itself is not a tensor and hence has no place in a covariant expression. All of these statements cannot be applied in metric-affine gravity for several reasons: first of all, the connections in metric-affine gravity are independent fields and, if they are not symmetric, there is the Cartan tensor that has to be constructed with linear combinations of the connection; secondly, transforming to a local inertial frame in metric-affine gravity is a two step process where we have to impose both that the metric is flat and that the connection vanishes; lastly, but more importantly, when inverting the procedure one must keep in mind that there could be dependencies on the connection in the equations other than those in the covariant derivatives; this means that the standard minimal coupling principle won't give, in general, the correct answer.
Let's now try to make the above discussion clear by using as an example the electromagnetic field. In order to get the hypermomentum of the magnetic field we have to start from the action

$$S_{EM} = -\frac{1}{4}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu}, \tag{2.54}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. In absence of gravity $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.55}$$

where A_μ is the electromagnetic four-potential. If we now follow the minimal coupling principle and simply replace the partial derivatives with the covariant ones we will get

$$F_{\mu\nu} = \bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - 2\Gamma^\sigma{}_{[\mu\nu]} A_\sigma \tag{2.56}$$

which can be easily shown to not be gauge invariant. Gauge invariance, however, is a critical aspect of the electromagnetich field since is both related to charge conservation and the fact that the magnetic field is a measurable quantity. This could lead to the explanation that electromagnetism is incompatible with the presence of torsion [15]. However the problem lies in something simpler which is, as we have said before, the assumption that the minimal coupling principle still works in metric-affine gravity. To demonstrate this point, let us focus our attention on the definition of the electromagnetic field as a differential form

$$\boldsymbol{F} \equiv \boldsymbol{dA}, \tag{2.57}$$

where $\boldsymbol{d}$ is the standard exterior derivative [16], which is closely related to Gauss theorem. Let us notice that the definition of the volume element has no depence on the connection, as in GR, and this implies that the definition of the external derivative when expressed in the form of partial derivatives should remain the same. On the other hand, partial derivatives are defined in the same way in both in this theory and GR, therefore $F_{\mu\nu}$ should be given in terms of partial derivatives by following the equation

$$F_{\mu\nu} \equiv dA = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.58}$$

This expression is not covariant, but it can be easily written in a covariant form

$$\begin{aligned} F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu =& \bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu + 2\Gamma^\sigma{}_{[\mu\nu]} A_\sigma \\ =& \bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu + 2S_{\mu\nu}{}^\sigma A_\sigma. \end{aligned} \tag{2.59}$$

This shows that the minimal coupling principle was leading us to the wrong expression. However this brings forth another problem, which is to find a new way to get the action in curved spacetime. What we can do is using the conjecture we've used before (which does not depend on the theory) and adapt it to express a metric-affine minimal coupling principle: the metric should be used in the matter action only for contracting indices and the connection should be used in order to construct the extra terms that we must to add in order to write a viable covariant

matter action. It can be then verified that the matter action of the electromagnetic field can be constructed by using this principle.
Now that we have a suitable expression for the electromagnetic field tensor we can proceed to derive the field equations. Since $F_{\mu\nu}$ has no dependence on the connection, we can instantly write

$$\Delta_\lambda^{\mu\nu} = 0. \tag{2.60}$$

The stress-energy tensor will have the standard form

$$T_{\mu\nu} = F_\mu{}^\sigma F_{\sigma\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \tag{2.61}$$

and the field equations will be

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \mathcal{K} F_\mu{}^\sigma F_{\sigma\nu} - \frac{\mathcal{K}}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \tag{2.62}$$

$$\bar{\nabla}_\lambda(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}) = 0. \tag{2.63}$$

We can now use the fact that the electromagnetic tensor is traceless, so we take the trace of 2.62 and we get

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 0, \tag{2.64}$$

which, as in the vacuum case, is an algebraic expression in $\mathcal{R}$ once that $f(\mathcal{R})$ is given. Solving it will give a numer of roots that we can call c_i and we will have that $f(c_i)$ will be constants. This implies, by using 2.63, that the metric is covariantly conserved by the covariant derivative defined using the connection, so we will have

$$\Gamma^\lambda{}_{\mu\nu} = \{{}^\lambda{}_{\mu\nu}\}, \tag{2.65}$$

and we will remain with the following field equation

$$\mathcal{R}_{\mu\nu} - \frac{1}{4} c_i g_{\mu\nu} = \mathcal{K}' F_\mu{}^\sigma F_{\sigma\nu} - \frac{\mathcal{K}'}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \tag{2.66}$$

which is the Einstein's equation with a cosmological constant and a modified coupling constant $\mathcal{K}'$, a result similar to that of the Palatini formalism.
As we mentioned earlier, a non vanishing $\Delta_\lambda^{\mu\nu}$ implies that there is no dependence on the connections in the matter action and we have found out that this is true for the electromagnetic field (and consequentely for any gaude field). The same is also true for a scalar field, since in that case the covariant derivatives simply reduce to partial ones. This means that neither of these fields will introduce torsion or non-metricity.

On the other hand we have the cases in which the hypermomentum does not vanish, like for the Dirac fields, but for the sake of simplicity we will leave out their description, which can be found in [14],[17] and [18].
To sum everything up, we have seen how, even if it's complicated by the presence of torsion and non-metricity, the metric-affine is the most general case of $f(R)$ gravity.

Chapter 3

Pulsar timing in $f(R)$ gravity with Yukawa potential

As we have said before, the simplest thing to do in order to modify GR is to generalize the Einstein-Hilbert Lagrangian by using an arbitrary function of the Ricci Scalar, with the condition that in the weak field limit we must again obtain GR in order to recover the costraints at the Solar system scale. As opposed to theories that add unknown particles or scalar fields to GR, for which then *ad-hoc* hypotheses have to be introduced for small scales, $f(R)$ gravity gets a modification to its potential from a Yukawa term, which is related to a lenght scale, and this should automatically avoid errors at small lenght scales.[19]

3.1 Post-Newtonian limit and Yukawa-like gravitational potential

Let us now look at the steps that lead us to the modification of the gravitational potential in the Post-Newtonian limit of $f(R)$ gravity[19]. First of all, as we have seen before, we start by modifying the Einstein-Hilbert action and we will do that by considering a generic fourth order action

$$\mathcal{A} = \int d^4x\sqrt{-g}[f(R) + \chi\mathcal{L}], \tag{3.1}$$

where $f(R)$ is an analytic function of the Ricci scalar, g is the determinant of the metric, $\chi = 16\pi G/c^4$ is the coupling constant and $\mathcal{L}$ is the standard fluid-matter Lagrangian. It can be easily seen that in the case $f(R) = R$ we get again GR. Now we vary the action with respect to the metric and we obtain the following field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\Box f'(R) = \frac{\chi}{2}T_{\mu\nu}, \tag{3.2}$$

where the semicolon is another notation for the covariant derivative. From this, we can get the trace

$$3\Box f'(R) + f'(R)R - 2f(R) = \frac{\chi}{2}T. \tag{3.3}$$

The next step is usually to make a conformal transformation from the Jordan to the Einstein frame [1] which turns the fourth order equations into second order ones, but, while the two frames are mathematically equivalent, the debate on if the two are physically equivalent is still going on. To be sure about the physical equivalence one should reproduce the results in both frames and then compare them. To avoid doing this, we will follow another route which is that of making the calculations in the Jordan frame (which means working with the fourth order equations) and consider the extra degrees of freedom as free parameters to be constrained by data. To compute the Post-Newtonian limit of $f(R)$ gravity we assume a general spherically symmetic metric

$$ds^2 = g_{tt}(x^0, r)(dx^0)^2 - g_{rr}(x^0, r)dr^2 - r^2 d\Omega^2, \tag{3.4}$$

where $x^0 = ct$ and $d\Omega$ is the solid angle. For the following computations we will set $c = 1$ and we will restore it later. Let us start by adding the perturbation to the metric tensor with respect to a Minkowskian background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and let us assume that the $f(R)$ Langrangian can be expanded in Taylor series

$$f(R) = \sum_n \frac{f^n(R_0)}{n!}(R - R_0)^n \simeq f_0 + f_0' R + f_0'' R^2 + f_0''' R^3 + \dots. \tag{3.5}$$

Once we have done that, we substitute the latter expression in the field equations and the trace and expand to the orders $\mathcal{O}(0), \mathcal{O}(2)$ and $\mathcal{O}(4)$. The resulting equations will be

$$f_0' r R^{(2)} - 2f_0' g^{(2)}_{tt,r} + 8f_0'' R^{(2)}_{,r} - f_0' r g^{(2)}_{tt,rr} + 4f_0'' r R^{(2)} = 0, \tag{3.6}$$

$$f_0' r R^{(2)} - 2f_0' g^{(2)}_{rr,r} + 8f_0'' R^{(2)}_{,r} - f_0' r g^{(2)}_{tt,rr} = 0, \tag{3.7}$$

$$2f_0' g_{rr} - r[f_0' r R^{(2)} - f_0' g^{(2)}_{tt,r} - f_0' g^{(2)}_{rr,r} + 4f_0'' R^{(2)}_{,r} + 4f_0'' r R^{(2)}_{,rr}] = 0, \tag{3.8}$$

[1] In the Jordan frame the scalar field multiplies the Ricci scalar, while in the Einstein frame the Ricci scalar appears alone.

$$f_0' r R^{(2)} + 6f_0''[2R^{(2)}_{,r} + rR^{(2)}_{,rr}] = 0, \tag{3.9}$$

$$2g^{(2)}_{rr} + r[2g^{(2)}_{tt,r} - rR^{(2)} + 2g^{(2)}_{rr,r} + rg^{(2)}_{tt,rr}] = 0. \tag{3.10}$$

where the comma is another notation for the standard derivative. If we use the trace equation 3.9, we can get the following general solution:

$$g^{(2)}_{tt} = \delta_0 - \frac{\delta_1}{f_0' r} + \frac{\delta_2(t)\lambda^2 e^{-r/\lambda}}{3} + \frac{\delta_3(t)\lambda^3 e^{r/\lambda}}{6r}, \tag{3.11}$$

$$g^{(2)}_{rr} = -\frac{\delta_1}{f_0' r} + \frac{\delta_2(t)(1 + r/\lambda)\lambda^2 e^{-r/\lambda}}{3r} + \frac{\delta_3(t)\lambda^3(1 - r/\lambda)e^{r/\lambda}}{6r}, \tag{3.12}$$

$$R^{(2)} = \delta_2(t)\frac{e^{-r/\lambda}}{r} + \frac{\delta_3 \lambda e^{r/\lambda}}{2r}, \tag{3.13}$$

where $\lambda = \sqrt{-6f_0''/f_0'}$, δ_0 can be ignored, δ_1 is an arbitrary constant, and δ_2 and δ_3 are arbitrary functions of time that, since the equations only include spatial derivatives, can be fixed to constant values. Finally, imposing that the metric must be asymptotically flat, we get

$$g_{tt}(x^0, r) = 1 - \frac{GM}{f_0' r} + \frac{\delta_2(t)\lambda^2 e^{-r/\lambda}}{3}, \tag{3.14}$$

$$g_{rr}(x^0, r) = 1 + \frac{GM}{f_0' r} + \frac{\delta_2(t)(1 + r/\lambda)\lambda^2 e^{-r/\lambda}}{3r}, \tag{3.15}$$

$$R^{(2)} = \frac{\delta_2(t) e^{-r/\lambda}}{r} \tag{3.16}$$

If we now remember that $g_{00} = 1 + 2\Phi_{grav} = 1 + g^{(2)}_{tt}$ we can extract the expression of the modified potential from 3.14 and 3.15

$$\Phi = -\frac{GM}{f_0' r} + \frac{\delta_2(t)\lambda^2 e^{-r/\lambda}}{6r}. \tag{3.17}$$

We can easily see that for $f(R) = R$ we recover the Newtonian potential, while this is not true for a generic $f(R)$. We can also rewrite the last expression in a more elegant form

$$\Phi = -\frac{GM}{r(1+\delta)}(1 + \delta e^{-r/\lambda}), \tag{3.18}$$

by defining $1 + \delta = f_0'$ and assuming δ_2 is quasiconstant and linked to δ by the following relation

$$\delta_2 = -\frac{6GM}{\lambda^2}\frac{\delta}{1+\delta}. \tag{3.19}$$

If $\delta = 0$ we recover the Newtonian potential, while if $\delta \neq 0$ we have a first term which is the gravitational potential due to a point-like mass and then a second term which is Yukawa-like with scale lenght λ. This is important for two major reasons: first, the scale lenght comes naturally from the theory, which means that no *ad hoc* correction must be introduced by hand; secondly, the second term being Yukawa-like can be used as a screening mechanism since it is negligible for small scales and only relevant at bigger scales like the extra-galactic one.
Finally, let us look at the line element ds^2. In ordere to obtain that, we can manipulate 3.14 and 3.15 and we will obtain

$$ds^2 = [1 + \Phi(r)]dt^2 - [1 - \Psi(r)]dr^2 - r^2 d\Omega^2, \tag{3.20}$$

where the potentials $\Phi(r)$ and $\Psi(r)$ are given by

$$\Phi(r) = -\frac{2GM(\delta e^{-\frac{r}{\lambda}} + 1)}{rc^2(\delta + 1)}, \tag{3.21}$$

$$\Psi(r) = \frac{2GM}{rc^2}\left[\frac{(\delta e^{-\frac{r}{\lambda}} + 1)}{(\delta + 1)} + \frac{(\frac{\delta r e^{-\frac{r}{\lambda}}}{\lambda} - 2)}{(\delta + 1)}\right]. \tag{3.22}$$

It can be easily seen that the potential $\Psi(r)$ can be written as

$$\Psi(r) = \Phi(r) + \delta\Phi(r), \tag{3.23}$$

where $\delta\Phi(r)$ is an extra contribution to the total gravitational potential. Since we are interested in stellar system scales, it can be shown that [19] $\Psi(r) \sim \Phi(r)$. So, in the end, we get that the line element can be written as

$$ds^2 = [1 + \Phi(r)]dt^2 - [1 - \Phi(r)]dr^2 - r^2 d\Omega^2. \tag{3.24}$$

3.2 Einstein delay in a Yukawa potential

Let us now look at the corrections in the Einstein delay caused by the added Yukawa-like term. Since the considerations that we made are just physical ones, and hence do not depend on the expression of the potential, we can start our computation from the same point as GR by saying that, in order to compute the time dependent part of the Einstein delay, we can use

$$\phi(\boldsymbol{x}) = -\frac{Gm_c(1 + e^{-r/\lambda})}{c^2|\boldsymbol{x} - \boldsymbol{x_c}|(1 + \delta)}. \tag{3.25}$$

and the weak field approximation. Starting from this, we can use the same step-by-step computation that we used for GR.
Firstly we will have that

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 1 - \frac{Gm_c(1+\delta e^{-\frac{|\boldsymbol{x_p}-\boldsymbol{x_c}|}{\lambda}})}{c^2|\boldsymbol{x_p}-\boldsymbol{x_c}|(1+\delta)} - \frac{v_p^2}{2c^2}, \tag{3.26}$$

in which we can substitute the pulsar velocity v_p from 1.73 in order to obtain

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 1 - \frac{G}{c^2(1+\delta)}\left[\frac{m_c(m_p+2m_c)}{m_p+m_c}\frac{1}{r}(1+\delta e^{-r/\lambda}) - \frac{m_c^2}{m_p+m_c}\frac{1}{2a}(1+\delta e^{2a/\lambda})\right]. \tag{3.27}$$

It is possible to separate the GR Einstein delay, but, for the sake of showing the full computation, we will do this at a later time. We now use the parametrization of Keplerian orbits in terms of the eccentric anomaly u

$$u - e\sin u = \frac{2\pi}{P_b}(t-t_0), \tag{3.28}$$

and, after differentiation and some manipulations, we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{2\pi}{P_b}\frac{1}{1-e\cos u}\frac{\mathrm{d}T}{\mathrm{d}u}. \tag{3.29}$$

This will lead us to

$$\frac{2\pi}{P_b}\left(\frac{\mathrm{d}T}{\mathrm{d}u}\right) = (1-e\cos u)\left[1 - \frac{G}{c^2(1+\delta)}\left[\frac{m_c(m_p+2m_c)}{m_p+m_c}\frac{1}{r}(1+\delta e^{-\frac{r}{\lambda}}) - \frac{m_c^2}{m_p+m_c}\frac{1}{2a}(1+\delta e^{-2a/\lambda})\right]\right]. \tag{3.30}$$

We can now rearrange the terms in order to get

$$\frac{2\pi}{P_b}\left(\frac{\mathrm{d}T}{\mathrm{d}u}\right) = (1-e\cos u)\times \times\left\{\left[1 - \frac{G}{c^2(1+\delta)}\left[\frac{m_c(m_p+2m_c)}{m_p+m_c}\frac{1}{r} - \frac{m_c^2}{m_p+m_c}\frac{1}{2a}\right]\right] - \frac{G}{c^2(1+\delta)}\left[\frac{m_c(m_p+2m_c)}{m_p+m_c}\frac{1}{r}\delta e^{-\frac{r}{\lambda}} - \frac{m_c^2}{m_p+m_c}\frac{1}{2a}\delta e^{-2a/\lambda}\right]\right\}. \tag{3.31}$$

Let us now focus on the correction term. Let us start by expanding the exponentials

$$
\begin{aligned}
&-\frac{G}{c^2(1+\delta)}\left[\frac{m_c(m_p+2m_c)}{m_p+m_c}\frac{1}{r}\delta e^{-\frac{r}{\lambda}}-\frac{m_c^2}{m_p+m_c}\frac{1}{2a}\delta e^{-2a/\lambda}\right]\\
&\simeq -\frac{G}{c^2(1+\delta)}\left[\frac{m_c(m_p+2m_c)}{m_p+m_c}\frac{1}{r}\delta(1-\frac{r}{\lambda}+\frac{r^2}{2\lambda^2})-\frac{m_c^2}{m_p+m_c}\frac{1}{2a}\delta(1-\frac{2a}{\lambda}+\frac{2a^2}{\lambda^2})\right]
\end{aligned}
\tag{3.32}
$$

If we now rearrange the terms, we will se that some of them will be the same as GR except for a δ factor, so they can be put together in order to simplify the $(+\delta)$ in the denominator. Then we substitute r by using

$$
r = a(1-e\cos u). \tag{3.33}
$$

We can then make some calculations to simplify the expression and the end result will be

$$
\begin{aligned}
\frac{2\pi}{P_b}\left(\frac{\mathrm{d}T}{\mathrm{d}u}\right) &= (1-e\cos u)\times\\
&\times\left[1-\frac{G}{c^2}\left[\frac{m_c(m_p+2m_c)}{m_p+m_c}\frac{1}{r}-\frac{m_c^2}{m_p+m_c}\frac{1}{2a}\right]\right]\\
&-\frac{G\delta}{c^2(1+\delta)}\left[\frac{m_c^2\left(\frac{2a^2}{\lambda^2}-\frac{2a}{\lambda}+1\right)\cos(u)}{2a(m_C+m_p)}\right.\\
&+\frac{m_c(2m_c+m_p)\left(\frac{a^2(1-e\cos(u))^2}{2\lambda^2}-\frac{a(1-e\cos(u))}{\lambda}\right)}{a(m_c+m_p)}\\
&\left.-\frac{m_c^2\left(\frac{2a^2}{\lambda^2}-\frac{2a}{\lambda}\right)}{2a(m_c+m_p)}\right]
\end{aligned}
\tag{3.34}
$$

We then use the same approximation that we used in 1.81, which will simplify the first term of the expression since it is the GR Eintein delay. Once again we can reabsorb the multiplying factor as a rescaling of proper time.This factor should now also appear in front of the correction term, but it can be easily seen, using a rough estimate, that the value of the factor is around one. Beacuse of this, we can also omit it. The result will then be

$$\left(\frac{dT}{du}\right) = \frac{P_b}{2\pi}(1-e\cos u)-\gamma\cos u-\frac{P_b}{2\pi}(1-e\cos u)-\frac{P_b}{2\pi}\frac{G\delta}{c^2(1+\delta)}\left[\frac{m_c^2\left(\frac{2a^2}{\lambda^2}-\frac{2a}{\lambda}+1\right)\cos(u)}{2a(m_C+m_p)}\right.$$
$$\left.+\frac{m_c(2m_c+m_p)\left(\frac{a^2(1-e\cos(u))^2}{2\lambda^2}-\frac{a(1-e\cos(u))}{\lambda}\right)}{a(m_c+m_p)}-\frac{m_c^2\left(\frac{2a^2}{\lambda^2}-\frac{2a}{\lambda}\right)}{2a(m_c+m_p)}\right]. \tag{3.35}$$

We proceed again to write

$$T = t - \Delta_E, \tag{3.36}$$

and, by using the fact that

$$\frac{2\pi}{P_b}\frac{\mathrm{d}t}{\mathrm{d}u} = 1 - e\cos u, \tag{3.37}$$

we will obtain

$$\left(\frac{\mathrm{d}\Delta_E}{\mathrm{d}u}\right) = \gamma\cos u + -\frac{P_b}{2\pi}\frac{G\delta}{c^2(1+\delta)}\left[\frac{m_c^2\left(\frac{2a^2}{\lambda^2}-\frac{2a}{\lambda}+1\right)\cos(u)}{2a(m_C+m_p)}\right.$$
$$\left.+\frac{m_c(2m_c+m_p)\left(\frac{a^2(1-e\cos(u))^2}{2\lambda^2}-\frac{a(1-e\cos(u))}{\lambda}\right)}{a(m_c+m_p)}-\frac{m_c^2\left(\frac{2a^2}{\lambda^2}-\frac{2a}{\lambda}\right)}{2a(m_c+m_p)}\right]. \tag{3.38}$$

All that is left to do now is to integrate the above expression. We can notice that the first term just gives back the GR Einstein delay , so the result will be the same as 1.104 which means taht it's not that difficult to compute. All that remains is to integrate the correction terms:

$$(\Delta_E)_{corr} = -\frac{\delta G m_c(\frac{1}{2}a^2e^2u(2m_c+m_p)+\frac{1}{4}a^2e^2(m_c+m_p)\sin 2u}{2ac^2(\delta+1)\lambda^2(m_c+m_p)}$$
$$-\frac{e\sin u(2a^2m_c+2a^2m_p-2a\lambda m_c-2a\lambda m_p-\lambda^2m_c)+au(am_p-2\lambda(m_c+m_p))}{2ac^2(\delta+1)\lambda^2(m_c+m_p)}. \tag{3.39}$$

Putting everything together we will finally have

$$\Delta_E = \gamma \sin u + -\frac{\delta G m_c(\frac{1}{2}a^2e^2u(2m_c+m_p)+\frac{1}{4}a^2e^2(m_c+m_p)\sin 2u}{2ac^2(\delta+1)\lambda^2(m_c+m_p)}$$
$$-\frac{e\sin u(2a^2m_c+2a^2m_p-2a\lambda m_c-2a\lambda m_p-\lambda^2 m_c)+au(am_p-2\lambda(m_c+m_p))}{2ac^2(\delta+1)\lambda^2(m_c+m_p)}. \tag{3.40}$$

As an example, we will show the values of the delay and the corrections for the Hulse and Taylor binary system (PSR1913+16), which can be computed by using the following data

Object	$a[m]$	$T[days]$	$m_c[M_\odot]$	$m_p[M_\odot]$	e
$PSR1913+16$	1.95×10^9	0.323	1.387	1.441	0.61

Table 3.1: The data used to make the computation have been taken from [20],[21].

$\Delta_{E,GR}[s]$	$\Delta_{E,f(R)}[s]$
2.566×10^{-4}	3.868×10^{-5}

Table 3.2: The Einstein time delay in General Relativity and the correction due our $f(R)$ theory. The parameter δ was set to 0.1

3.3 Shapiro delay in a Yukawa potential

In order to give a rough estimate of the correction to the Shapiro delay caused by the introduction of a Yukawa-like potential, we will use the same approach as 1.1.2. The only difference will be in the interpretation of Figure 1.2, since now the binary system will be in place of the Earth-Sun system and the pulsar will be substituted by the Earth. So let's start again from

$$\Delta_S = -\frac{2}{c}\int_{r_{obs}}^{r_p} |d\boldsymbol{x}|\phi(\boldsymbol{x}), \tag{3.41}$$

which becomes, after the substitution of the potential,

$$\Delta_S = \frac{2Gm_c}{c^3(1+\delta)}\int_0^d \frac{(1+\delta e^{-r/\lambda})}{r}d\rho \tag{3.42}$$

We then expand the exponential and we obtain

$$\Delta_S = \frac{2Gm_c}{c^3(1+\delta)} \int_0^d \frac{(1+\delta(1-\frac{r}{\lambda}+\frac{r^2}{2\lambda^2}))}{r} d\rho. \tag{3.43}$$

Once we have done that we can write r as

$$r = a\sqrt{u^2+1+2u\cos\theta}, \tag{3.44}$$

which is the same of 1.16, with the difference that now the distance Earth-Sun will be replaced by the semi-major axis of the binary system. It's now useful to write all of the resulting terms in order to make some remarks

$$\Delta_S = \frac{2Gm_c}{c^3(1+\delta)} \int_0^{\bar{u}} \frac{du}{\sqrt{(u^2+1+2u\cos\theta)}} + \frac{2Gm_c\delta}{c^3(1+\delta)} \int_0^{\bar{u}} \frac{du}{\sqrt{(u^2+1+2u\cos\theta)}} - \frac{2Gm_c\delta}{c^3(1+\delta)} \int_0^{\bar{u}} \frac{a}{\lambda} du + \frac{2Gm_c\delta}{c^3(1+\delta)} \int_0^{\bar{u}} \frac{a^2\sqrt{(u^2+1+2u\cos\theta)}}{2\lambda^2} du. \tag{3.45}$$

It can be easily seen that the first two terms can be put together in order to get the GR Shapiro delay, so we will now focus on the two remaining correction terms. All that it's left to do is to integrate them, so we will get

$$(\Delta_S)_{corr} = \frac{2dGm_c\delta}{c^3(1+\delta)\lambda} - \frac{1}{2}\frac{a^2Gm_c\delta(\cos\theta + \log(1+\cos\theta)\sin^2\theta)}{c^3(1+\delta)\lambda^2} + \frac{1}{2}\frac{a^2Gm_c\delta\left((\frac{d}{a}+\cos\theta)\sqrt{1+\frac{d^2}{a^2}+\frac{2d\cos\theta}{a}} + \log\left(\frac{d}{a}+\cos\theta+\sqrt{1+\frac{d^2}{a^2}+\frac{2d\cos\theta}{a}}\right)\right)}{c^3(1+\delta)\lambda^2}. \tag{3.46}$$

In the end, putting everything together, we will get

$$\Delta_S = \frac{2GM_\odot}{c^3}\left[\log\left(\frac{d}{r_{es}}\right) - \log\left(\frac{1+\cos\theta}{2}\right)\right] + \frac{2dGm_c\delta}{c^3(1+\delta)\lambda} - \frac{1}{2}\frac{a^2Gm_c\delta(\cos\theta + \log(1+\cos\theta)\sin^2\theta)}{c^3(1+\delta)\lambda^2} + \frac{1}{2}\frac{a^2Gm_c\delta\left((\frac{d}{a}+\cos\theta)\sqrt{1+\frac{d^2}{a^2}+\frac{2d\cos\theta}{a}} + \log\left(\frac{d}{a}+\cos\theta+\sqrt{1+\frac{d^2}{a^2}+\frac{2d\cos\theta}{a}}\right)\right)}{c^3(1+\delta)\lambda^2}. \tag{3.47}$$

As we have done before, we present an example calculated for the Hulse and Taylor binary system

Object	$a[m]$	$d[m]$	$m_c[M_\odot]$	$\Delta_{S,GR}[s]$	$\Delta_{S,1}[s]$	$\Delta_{S,2}[s]$
$PSR1913+16$	1.95×10^9	1.97×10^{20}	1.387	2.566×10^{-4}	1.332×10^{-8}	4.496×10^{-12}

Table 3.3: The Shapiro time delay in General Relativity with the first and second order corrections due to $f(R)$ theory. The data used to make the computation have been taken from [20],[21]. The parameter δ was set to 0.1

Conclusions

We have seen that the General Relativistic effects on pulsar signals are very tangible and result in small delays in the arrival times to the observer. More than that, we have seen that using $f(R)$ theories leads to corrections that, even if small, are still present. This is very important because the Einstein and Shapiro delays formulas contain three of the five Post Keplerian parameters and, since these parameters are involved in the determination of some characteristics of the binary systems, a better accuracy of the formulas and, as a consequence, of the parameters can lead to a more accurate estimate of said characteristics. Going forward, the foundations have been laid to calculate said corrections of the post keplerian parameters and also to get a better estimate of the pulsar signals' times of arrival.

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