

# GammaLib Maths

## Mathematical Implementation

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# 1 Introduction

## 2 Functions

### 2.1 King profile

The King Profile is a commonly used parametrisation of instrument PSFs in astronomy. It is radially symmetric and compared to a simple Gaussian, it allows longer tails in the distribution of events from a point source. The probability density function is defined as follows:

$$P(r|\sigma, \gamma) = \frac{1}{2\pi\sigma^2} \left(1 - \frac{1}{\gamma}\right) \left(1 + \frac{1}{2\gamma} \frac{r^2}{\sigma^2}\right)^{-\gamma} \quad (1)$$

To integrate the function we substitute

$$u = \frac{r^2}{2\sigma^2} \quad (2)$$

which gives

$$rdr = \sigma^2 du \quad (3)$$

and hence

$$\frac{1}{2\pi\sigma^2} \left(1 - \frac{1}{\gamma}\right) \left(1 + \frac{1}{2\gamma} \frac{r^2}{\sigma^2}\right)^{-\gamma} rdr = \frac{1}{2\pi\sigma^2} \left(1 - \frac{1}{\gamma}\right) \left(1 + \frac{u}{\gamma}\right)^{-\gamma} \sigma^2 du \quad (4)$$

The integral is thus given by

$$\begin{aligned} \int_0^{r_{\max}} \int_0^{2\pi} P(r|\sigma, \gamma) r dr d\phi &= \left(1 - \frac{1}{\gamma}\right) \int_0^{u_{\max}} \left(1 + \frac{u}{\gamma}\right)^{-\gamma} du \\ &= \left(1 - \frac{1}{\gamma}\right) \left( \frac{(\gamma + u) \left(\frac{\gamma + u}{\gamma}\right)^{-\gamma}}{1 - \gamma} \right) \Bigg|_0^{u_{\max}} \\ &= \left(\frac{\gamma - 1}{\gamma}\right) \left( \frac{(\gamma + u) \left(\frac{\gamma + u}{\gamma}\right)^{-\gamma}}{1 - \gamma} \right) \Bigg|_0^{u_{\max}} \\ &= - \left( \frac{(\gamma + u) \left(\frac{\gamma + u}{\gamma}\right)^{-\gamma}}{\gamma} \right) \Bigg|_0^{u_{\max}} \\ &= - \left( \frac{\gamma + u}{\gamma} \right)^{1-\gamma} \Bigg|_0^{u_{\max}} \\ &= - \left( 1 + \frac{u}{\gamma} \right)^{1-\gamma} \Bigg|_0^{u_{\max}} \\ &= - \left( 1 + \frac{u_{\max}}{\gamma} \right)^{1-\gamma} + 1 \\ &= 1 - \left( 1 + \frac{u_{\max}}{\gamma} \right)^{1-\gamma} \end{aligned} \quad (5)$$

where

$$u_{\max} = \frac{r_{\max}^2}{2\sigma^2} \quad (6)$$

To determine the value of  $u_{\max}$  that corresponds to a given containment fraction  $F$ , the equation

$$1 - \left(1 + \frac{u_{\max}}{\gamma}\right)^{1-\gamma} = F \quad (7)$$

has to be solved for  $u_{\max}$ , resulting in

$$u_{\max} = \left((1 - F)^{\frac{1}{1-\gamma}} - 1\right) \gamma \quad (8)$$

This can be converted into  $r_{\max}$  using

$$r_{\max} = \sigma \sqrt{2u_{\max}} \quad (9)$$

## 3 Spatial models

### 3.1 ShellFunction

#### 3.1.1 Small angle approximation

The shell function is a radial function  $f(\theta)$ , where  $\theta$  is the angular separation between shell centre and the actual location. In the small angle approximation  $\sin \theta \approx \theta$ , the shell function is given by

$$f(\theta) = f_0 \begin{cases} \sqrt{\theta_{\text{out}}^2 - \theta^2} - \sqrt{\theta_{\text{in}}^2 - \theta^2} & \text{if } \theta \leq \theta_{\text{in}} \\ \sqrt{\theta_{\text{out}}^2 - \theta^2} & \text{if } \theta_{\text{in}} < \theta \leq \theta_{\text{out}} \\ 0 & \text{if } \theta > \theta_{\text{out}} \end{cases} \quad (10)$$

is the radial function.  $f_0$  is a normalization constant that is determined by

$$2\pi \int_0^{\pi/2} f(\theta) \theta \, d\theta = 1 \quad (11)$$

in the small angle approximation. Using

$$\int x \sqrt{a - x^2} \, dx = -\frac{1}{3}(a - x^2)^{3/2} \quad (12)$$

we obtain

$$\begin{aligned} \frac{1}{f_0} &= -\frac{2\pi}{3} \left[ (\theta_{\text{out}}^2 - \theta_{\text{in}}^2)^{3/2} - (\theta_{\text{in}}^2 - \theta_{\text{in}}^2)^{3/2} - (\theta_{\text{out}}^2)^{3/2} + (\theta_{\text{in}}^2)^{3/2} + (\theta_{\text{out}}^2 - \theta_{\text{out}}^2)^{3/2} - (\theta_{\text{out}}^2 - \theta_{\text{in}}^2)^{3/2} \right] 3 \\ &= -\frac{2\pi}{3} \left[ (\theta_{\text{out}}^2 - \theta_{\text{in}}^2)^{3/2} - (\theta_{\text{out}}^2)^{3/2} + (\theta_{\text{in}}^2)^{3/2} - (\theta_{\text{out}}^2 - \theta_{\text{in}}^2)^{3/2} \right] \end{aligned} \quad (14)$$

$$= -\frac{2\pi}{3} \left[ -(\theta_{\text{out}}^2)^{3/2} + (\theta_{\text{in}}^2)^{3/2} \right] \quad (15)$$

$$= \frac{2\pi}{3} \left[ \theta_{\text{out}}^3 - \theta_{\text{in}}^3 \right] \quad (16)$$

### 3.1.2 Spherical formulation

The shell function on a sphere is given by

$$f(\theta) = f_0 \begin{cases} \sqrt{\sin^2 \theta_{\text{out}} - \sin^2 \theta} - \sqrt{\sin^2 \theta_{\text{in}} - \sin^2 \theta} & \text{if } \theta \leq \theta_{\text{in}} \\ \sqrt{\sin^2 \theta_{\text{out}} - \sin^2 \theta} & \text{if } \theta_{\text{in}} < \theta \leq \theta_{\text{out}} \\ 0 & \text{if } \theta > \theta_{\text{out}} \end{cases} \quad (17)$$

The normalization constant  $f_0$  is determined by

$$2\pi \int_0^{\pi/2} f(\theta) \sin \theta \, d\theta = 1 \quad (18)$$

Using

$$\int \sin x \sqrt{a - \sin^2 x} \, dx = -\frac{\cos x \sqrt{\cos 2x + 2a - 1}}{2\sqrt{2}} - \frac{a-1}{2} \ln \left( \sqrt{2} \cos x + \sqrt{\cos 2x + 2a - 1} \right) \quad (19)$$

and

$$2a = 2 \sin^2 \theta_0 = 1 - \cos 2\theta_0 \quad (20)$$

we can write

$$\int \sin x \sqrt{\sin^2 \theta_0 - \sin^2 x} \, dx = -\frac{\cos x \sqrt{\cos 2x - \cos 2\theta_0}}{2\sqrt{2}} + \frac{\cos 2\theta_0 + 1}{4} \ln \left( \sqrt{2} \cos x + \sqrt{\cos 2x - \cos 2\theta_0} \right) \quad (21)$$

For the special case of  $x = 0$ , the integral becomes

$$-\frac{\sqrt{1 - \cos 2\theta_0}}{2\sqrt{2}} + \frac{\cos 2\theta_0 + 1}{4} \ln \left( \sqrt{2} + \sqrt{1 - \cos 2\theta_0} \right) \quad (22)$$

while for the special case  $x = \theta_0$ , it reduces to

$$\frac{\cos 2\theta_0 + 1}{4} \ln \left( \sqrt{2} \cos \theta_0 \right) \quad (23)$$

Using these equations, we can compute the required integrals:

$$I_1 = \int_0^{\theta_{\text{in}}} \sin \theta \sqrt{\sin^2 \theta_{\text{out}} - \sin^2 \theta} \, d\theta = \frac{\sqrt{1 - \cos 2\theta_{\text{out}}} - \cos \theta_{\text{in}} \sqrt{\cos 2\theta_{\text{in}} - \cos 2\theta_{\text{out}}}}{2\sqrt{2}} + \frac{\cos 2\theta_{\text{out}} + 1}{4} \ln \left( \frac{\sqrt{2} \cos \theta_{\text{in}} + \sqrt{\cos 2\theta_{\text{in}} - \cos 2\theta_{\text{out}}}}{\sqrt{2} + \sqrt{1 - \cos 2\theta_{\text{out}}}} \right) \quad (24)$$

$$I_2 = \int_0^{\theta_{\text{in}}} \sin \theta \sqrt{\sin^2 \theta_{\text{in}} - \sin^2 \theta} \, d\theta = \frac{\sqrt{1 - \cos 2\theta_{\text{in}}}}{2\sqrt{2}} + \frac{\cos 2\theta_{\text{in}} + 1}{4} \ln \left( \frac{\sqrt{2} \cos \theta_{\text{in}}}{\sqrt{2} + \sqrt{1 - \cos 2\theta_{\text{in}}}} \right) \quad (25)$$

$$I_3 = \int_{\theta_{\text{in}}}^{\theta_{\text{out}}} \sin \theta \sqrt{\sin^2 \theta_{\text{out}} - \sin^2 \theta} \, d\theta = \frac{\cos \theta_{\text{in}} \sqrt{\cos 2\theta_{\text{in}} - \cos 2\theta_{\text{out}}}}{2\sqrt{2}} + \frac{\cos 2\theta_{\text{out}} + 1}{4} \ln \left( \frac{\sqrt{2} \cos \theta_{\text{out}}}{\sqrt{2} \cos \theta_{\text{in}} + \sqrt{\cos 2\theta_{\text{in}} - \cos 2\theta_{\text{out}}}} \right) \quad (26)$$

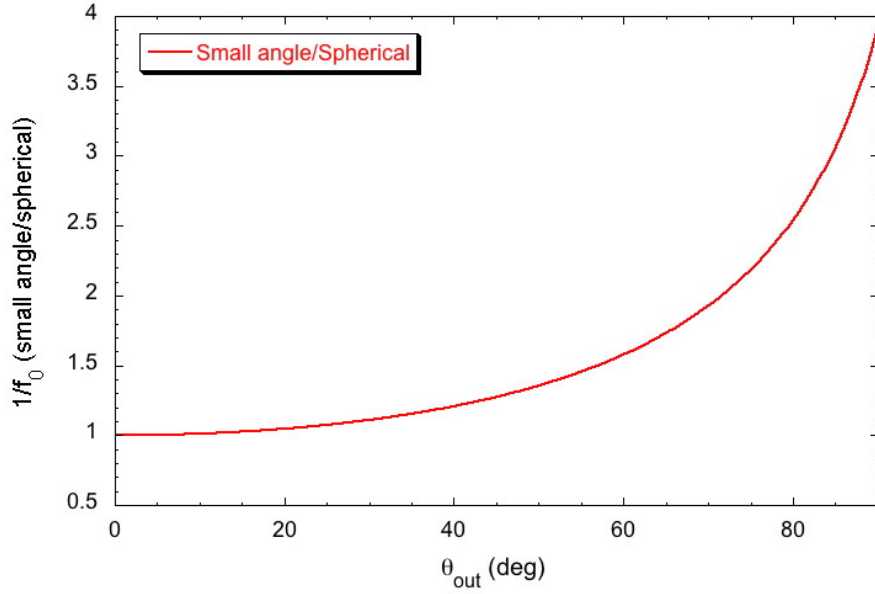


Figure 1: Ratio in  $\frac{1}{f_0}$  between the small angle approximation and the correct spherical computation as function of  $\theta_{\text{out}}$  for  $\theta_{\text{in}} = \frac{2}{3}\theta_{\text{out}}$ .

We thus obtain

$$\begin{aligned}
 \frac{1}{2\pi f_0} &= I_1 - I_2 + I_3 \\
 &= \frac{\sqrt{1 - \cos 2\theta_{\text{out}}} - \sqrt{1 - \cos 2\theta_{\text{in}}}}{2\sqrt{2}} \\
 &+ \frac{\cos 2\theta_{\text{out}} + 1}{4} \ln \left( \frac{\sqrt{2} \cos \theta_{\text{out}}}{\sqrt{2} + \sqrt{1 - \cos 2\theta_{\text{out}}}} \right) - \frac{\cos 2\theta_{\text{in}} + 1}{4} \ln \left( \frac{\sqrt{2} \cos \theta_{\text{in}}}{\sqrt{2} + \sqrt{1 - \cos 2\theta_{\text{in}}}} \right) \quad (27)
 \end{aligned}$$

The ratio in  $\frac{1}{f_0}$  between the small angle approximation and the correct spherical computation is shown in Fig. 1 as function of  $\theta_{\text{out}}$  for  $\theta_{\text{in}} = \frac{2}{3}\theta_{\text{out}}$ .

## 4 Spectral models

### 4.1 PowerLaw

The power law spectral model is defined by

$$I(E) = k \left( \frac{E}{p} \right)^\gamma \quad (28)$$

where

- $k$  is the normalization of the power law (units:  $\text{ph cm}^{-2} \text{s}^{-1} \text{MeV}^{-1}$ ),
- $p$  is the pivot energy (units: MeV), and
- $\gamma$  is the spectral index (which is usually negative).

Each of the 3 parameters is factorised into a scaling factor and a value, i.e.  $k = k_s k_v$ ,  $p = p_s p_v$ , and  $\gamma = \gamma_s \gamma_v$ . The `GModelSpectralPlaw::eval_gradients` returns the gradients with respect to the parameter value of the factorisation. Note that for any parameter  $a = a_s a_v$ :

$$\frac{\delta I}{\delta a_v} = \frac{\delta I}{\delta a} \frac{\delta a}{\delta a_v} = \frac{\delta I}{\delta a} a_s \quad (29)$$

The parameter value gradients for the power law are given by

$$\frac{\delta I}{\delta k_v} = k_s \left( \frac{E}{p} \right)^\gamma = \frac{I(E)}{k_v} \quad (30)$$

$$\frac{\delta I}{\delta p_v} = -\frac{\gamma}{p_v} k \left( \frac{E}{p} \right)^\gamma = -\frac{\gamma}{p_v} I(E) \quad (31)$$

$$\frac{\delta I}{\delta \gamma_v} = \gamma_s \ln \left( \frac{E}{p} \right) k \left( \frac{E}{p} \right)^\gamma = \gamma_s \ln \left( \frac{E}{p} \right) I(E) \quad (32)$$

## 4.2 PowerLaw2

This flavour of the power law spectral model uses the integral flux  $f$  within the energy range  $E_{\min}$  and  $E_{\max}$  as free parameter instead of the normalization  $k$ . The integral flux  $f$  is given by

$$f = \int_{E_{\min}}^{E_{\max}} I(E) dE \quad (33)$$

The power law model is then defined by

$$I(E) = \tilde{k} E^\gamma \quad (34)$$

where

$$\tilde{k} = \begin{cases} \frac{f}{\ln E_{\max} - \ln E_{\min}} & \text{if } \gamma = -1 \\ \frac{f(1 + \gamma)}{E_{\max}^{\gamma+1} - E_{\min}^{\gamma+1}} & \text{else} \end{cases} \quad (35)$$

is obtained from Eq. (33).

Each of the 4 parameters is factorised into a scaling factor and a value, e.g.  $f = f_s f_v$  and  $\gamma = \gamma_s \gamma_v$ . It is assumed that  $E_{\min}$  and  $E_{\max}$  are fixed parameters, and `GModelSpectralPlaw2::eval_gradients` returns valid gradients only for  $f_v$  and  $\gamma_v$ .

The flux value gradient  $f_v$  is given by

$$\frac{\delta I}{\delta f_v} = \frac{\delta I}{\delta f} \frac{\delta f}{\delta f_v} = \frac{\delta I}{\delta f} f_s = \frac{\delta \tilde{k}}{\delta f} E^\gamma f_s = \frac{\tilde{k}}{f} E^\gamma f_s = \frac{I(E)}{f} f_s = \frac{I(E)}{f_v} \quad (36)$$

The index value gradient  $\gamma_v$  is given by

$$\frac{\delta I}{\delta \gamma_v} = \frac{\delta I}{\delta \gamma} \frac{\delta \gamma}{\delta \gamma_v} = \frac{\delta I}{\delta \gamma} \gamma_s = \left( \frac{\delta \tilde{k}}{\delta \gamma} E^\gamma + \tilde{k} E^\gamma \ln E \right) \gamma_s = \left( \frac{1}{\tilde{k}} \frac{\delta \tilde{k}}{\delta \gamma} + \ln E \right) \tilde{k} E^\gamma \gamma_s = \left( \frac{1}{\tilde{k}} \frac{\delta \tilde{k}}{\delta \gamma} + \ln E \right) I(E) \gamma_s \quad (37)$$

where

$$\frac{\delta \tilde{k}}{\delta \gamma} = \begin{cases} 0 & \text{if } \gamma = -1 \\ \frac{f \left( E_{\max}^{\gamma+1} - E_{\min}^{\gamma+1} \right) - f(1 + \gamma) \left( E_{\max}^{\gamma+1} \ln E_{\max} - E_{\min}^{\gamma+1} \ln E_{\min} \right)}{\left( E_{\max}^{\gamma+1} - E_{\min}^{\gamma+1} \right)^2} & \text{else} \end{cases} \quad (38)$$



Note that for  $\gamma \neq -1$

$$\frac{1}{\tilde{k}} \frac{\delta \tilde{k}}{\delta \gamma} = \frac{1}{1 + \gamma} - \frac{\left( E_{\max}^{\gamma+1} \ln E_{\max} - E_{\min}^{\gamma+1} \ln E_{\min} \right)}{\left( E_{\max}^{\gamma+1} - E_{\min}^{\gamma+1} \right)} \quad (39)$$